Variational Principle Approach to General Relativity

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The candidate has passed oral examination by members of examination panel. This report has been accepted by the panel as partially fulfilment of the course 261493 Independent Study.

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Title: Variational Principle Approach to General Relativity

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Abstract

General relativity theory is a theory for gravity which Galilean relativity fails to explain. Variational principle is a method which is powerful in physics. All physical laws is believed that they can be derived from action using variational principle. Einstein's field equation, which is essential law in general relativity, can also be derived using this method. In this report we show derivation of the Einstein's field equation using this method. We also extend the gravitational action to include boundary terms and to obtain Israel junction condition on hypersurface. The method is powerful and is applied widely to braneworld gravitational theory.

Contents

Chapter 1

Introduction

1.1 Background

Classical mechanics is useful to explain physical phenomena but it fails to explain gravity. General relativity (GR) is proposed to be a satisfactory theory for gravity. Studying GR leads to the Einstein's field equation which can be derived in standard way. Our interest is to apply variation principle to derive the field equation in GR.

1.2 Objectives

To study general relativity and tensor calculus which is applied to derive, with variational principle, Einstein's field equation and Israel junction condition.

1.3 Frameworks

- To explain failure of Newtonian mechanics, special relativity.
- To study general relativity.
- To use variational principle for Einstein's field equation and for Israel junction condition.

1.4 Expected Use

- To obtain Einstein's field equation from variational method and to obtain junction condition by including of surface term in action.
- A derived-in-detailed report for those who interest with thoroughly calculation from variational method in general relativity.
- Attaining understanding of concept of general relativity and having skills on tensor calculus.

1.5 Tools

- Text books in physics and mathematics.
- A high efficiency personal computer.
- Software e.g. LaTeX, WinEdit and Photoshop

1.6 Procedure

- Studying special relativity.
- Studying tensor analysis and calculation skills.
- Studying concepts of general relativity.
- Studying Einstein's field equation by evaluating coupling constant.
- Studying Einstein's field equation by variational principle.
- Including surface term in action and deriving junction condition.
- Making conclusion and preparing report and other presentation.

1.7 Outcome

- Understanding of basic ideas of classical mechanics, special relativity and general relativity.
- Attaining skills of tensor calculation.
- Understanding in detail of the variation method in general relativity.

Chapter 2

Failure of classical mechanics and introduction to special relativity

2.1 Inertial reference frames

Newton introduced his three laws of motion as axioms of classical mechanics. These laws have successfully explained motion of most objects known to us. The Newton's first law states that *a body remains at rest or in uniform motion*. This law introduces a frame of reference called inertial frame. All proceeded dynamical laws base on this law. But what and where is it?

To know or to measure velocity of a particle, we need a frame of reference. For example when we measure speed of a car, a particular spot on ground is inertial frame. The ground is on the Earth and is not really inertial frame due to gravity. Furthermore, all stars in the universe possess gravity therefore nowhere is really locate on true inertial frame! However we will discuss about inertial frame again in chapter 3.

2.2 Failure of Galilean transformation

In Newtonian mechanics, the concept of time and space are completely separable. Furthermore time is assumed to be absolute quantity and independent of any observers. Consider inertial frames of reference moving with constant velocity to each other. In classical mechanics there is transformation law between two inertial frames.

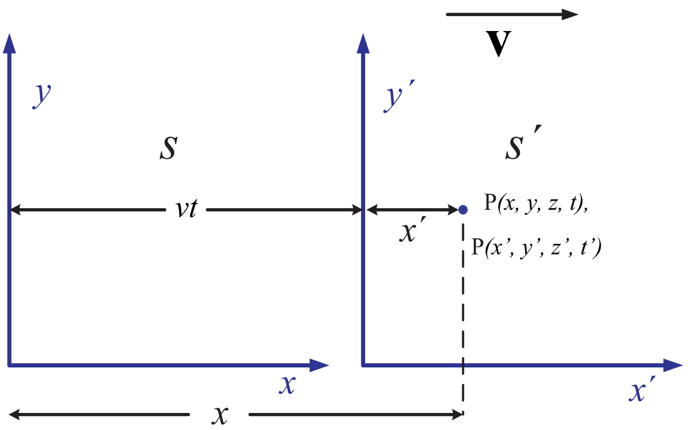

Figure 2.1: Relative velocity of two inertial frames

That is so called Galilean transformation:

$$\begin{aligned} x' &= x - vt \\ y' &= y \\ z' &= z \\ t' &= t. \end{aligned} \tag{2.1}$$

Newton's laws are invariant with respect to Galilean transformation

$$F_i = m\ddot{x}_i = m\ddot{x}'_i = F'_i \tag{2.2}$$

The principle of Galilean transformation yields that the velocity of light in two different inertial frames is measured with different values

$$u' = c - u \ \text{ and } \ u = c. \tag{2.3}$$

However the Galilean transformation fails to explain electromagnetic wave equation; electromagnetic wave equation is not invariant under Galilean transformation. Consider electromagnetic wave equation:

$$\nabla^2 \phi = \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2}$$

or

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t^2} = 0. \tag{2.4}$$

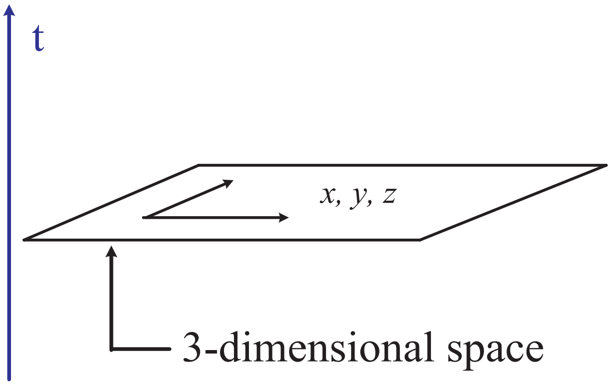

Figure 2.2: Illustration of four-dimensional spacetime, time axis is orthogonal to all three-dimensional spatial axes

Using chain rule and equation (2.1) to transform coordinate, the wave equation becomes

$$\frac{c^2 - v^2}{c^2}\frac{\partial^2\phi}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2\phi}{\partial t'\partial x'} + \frac{\partial^2\phi}{\partial y'^2} + \frac{\partial^2\phi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2\phi}{\partial t'^2} = 0. \quad (2.5)$$

This equation contradicts to Einstein's postulates in special relativity that physical laws should be the same in all inertial frames. Therefore we require new transformation law, **Lorentz transformation** which we shall discuss in the next section.

2.3 Introduction to special relativity

In framework of Newtonian mechanics we consider only three-dimensional space and flat geometry while in special relativity we consider space and time as one single entity called spacetime. We do not separate time from space and we have four dimensions of spacetime. Considering four-dimensional spacetime, time axis is orthogonal to all axes of space $(x,\ y,\ z)$.

The mathematical quantity which represents space and time is components of spacetime coordinates

$$\begin{aligned} x^a &= (x^0, x^1, x^2, x^3) \\ &= (ct, x, y, z) \qquad \text{in Cartesian coordinate} \end{aligned} \quad (2.6)$$

Special Relativity (SR) is a theory for physics in flat spacetime called Minkowski spacetime. If we are talking about spacetime, we must have *events*. Any events

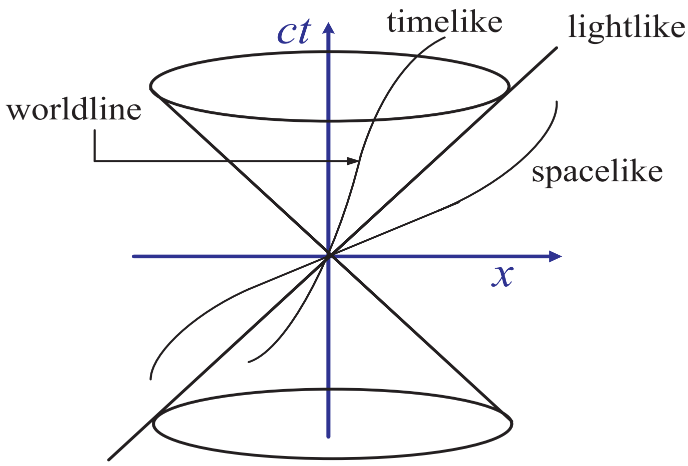

Figure 2.3: Lightcone and trajectory of a particle in four-dimensional spacetime

possess four coordinates describing where and when the particle is. Trajectories of particles in four-dimensional spacetime form a set of particles' *worldline*. If we reduce to one dimensional space, we will have two spatial dimensions left. Plotting these two spatial coordinate axes versus time, we have *spacetime diagram* as shown in Fig. 2.2

The electromagnetic wave equation introduces speed of light in vacuum which is $c = 1/\sqrt{\mu_0\epsilon_0}$, where μ_0 and ϵ_0 are permeability and permittivity of free space respectively. The value c is independent to all reference frame. This statement is confirmed by Michelson-Morley experiment. Einstein's principle of special relativity states that

- *The law of physical phenomena are the same in all inertial reference frames.*
- *The velocity of light is the same in all inertial reference frames.*

The spacetime interval or line-element in four dimensional spacetime is

$$\mathrm{d}s^2 = -c^2\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2. \tag{2.7}$$

We can see that if $\mathrm{d}s^2 = 0$, this equation becomes equation of spherical wave of light with radius $c\mathrm{d}t$. We can classify four-vectors into 3 classes namely spacelike, lightlike and timelike vectors. Any line elements are called

$$\begin{array}{rlrcl} \text{spacelike} & \text{if they lie in region} & \mathrm{d}s^2 & < & 0 \\ \text{lightlike or null} & \text{if they lie in region} & \mathrm{d}s^2 & = & 0 \\ \text{timelike} & \text{if they lie in region} & \mathrm{d}s^2 & > & 0. \end{array}$$

equation (2.7) can be expressed in general form as

$$ds^2 = dx^a dx_a = \sum_{a,b=0}^{3} g_{ab} dx^a dx^b = g_{ab} dx^a dx^b. \tag{2.8}$$

We use Einstein's summation convention. In this convention we do not need to use summation symbol in equation (2.8). If upper indies and lower indies of four-vectors are similar (repeated) in any terms, it implies sum over that indices. The symmetrical tensor g_{ab} is called *metric tensor*. When considering flat Minkowski spacetime, we write η_{ab} instead of g_{ab},

$$g_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.9}$$

The line element can be computed in matrix form,

$$\begin{aligned} ds^2 &= \left[\begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \right]^{\mathrm{T}} \begin{pmatrix} dx^0 \\ dx^1 \\ dx^2 \\ dx^3 \end{pmatrix} \\ &= -(dx^2)^2 + (dx^1)^2 + (dx^2)^2 + (dx^3)^2 \\ &= -c dt^2 + dx^2 + dy^2 + dz^2. \end{aligned} \tag{2.10}$$

Coordinate transformation law in SR is *Lorentz transformation*, a translation or boosts in one direction between two inertial frames moving relative to each other with constant velocity.

In relativity, time is not an absolute quantity. Time measured in each reference frame are not equal. Time measured in one inertial frame by observer on that frame is called *proper time*. Since speed of light is the same in all inertial frames, then the transformation law between two frames is written as

$$\begin{aligned} c dt' &= \gamma(dt - v dx/c) \\ dx' &= \gamma(dx - v dt) \\ dy' &= dy \\ dz' &= dz. \end{aligned} \tag{2.11}$$

This set of equations is Lorentz transformation. The symbol γ is Lorentz factor,

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}. \tag{2.12}$$

Lorentz transformation can also be written in general form,

$$\mathrm{d}x^a = \Lambda^a{}_b \mathrm{d}x^b. \tag{2.13}$$

where $\Lambda^a{}_b$ is Lorentz matrix,

$$\Lambda^a{}_b = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.14}$$

Here we define[1] $\beta \equiv v/c$ and $\gamma \equiv 1/\sqrt{1-\beta^2}$. Therefore equation (2.13) is

$$\begin{pmatrix} c\mathrm{d}t' \\ \mathrm{d}x' \\ \mathrm{d}y' \\ \mathrm{d}z' \end{pmatrix} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c\mathrm{d}t \\ \mathrm{d}x \\ \mathrm{d}y \\ \mathrm{d}z \end{pmatrix}. \tag{2.15}$$

Using equation (2.10) and equation (2.11), we obtain

$$\begin{aligned} -c^2\mathrm{d}t^2 + \mathrm{d}r^2 &= -c^2\mathrm{d}t'^2 + \mathrm{d}r'^2 \\ \text{or} \qquad -c^2\mathrm{d}t^2 + \mathrm{d}x^2 &= -c^2\mathrm{d}t'^2 + \mathrm{d}x'^2 \qquad (2.16) \\ \text{Therefore} \qquad \mathrm{d}s^2 &= \mathrm{d}s'^2 \qquad (2.17) \end{aligned}$$

where $\mathrm{d}r^2 = \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2$ and $\mathrm{d}r'^2 = \mathrm{d}x'^2 + \mathrm{d}y'^2 + \mathrm{d}z'^2$. Equation (2.16) is rotational transformation in (x, ct) space. Values of β and γ range in $0 \leq \beta \leq 1$ and $1 \leq \gamma \leq \infty$ respectively. If we introduce new parameter ω and write β and γ in term of this new variables as

$$\begin{aligned} \beta &= \tanh\omega \\ \gamma &= \cosh\omega, \end{aligned} \tag{2.18}$$

where ω in the angle of rotation in this (x, ct) space, then Lorentz transformation becomes

$$\begin{pmatrix} ct' \\ x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cosh\omega & -\sinh\omega & 0 & 0 \\ -\sinh\omega & \cosh\omega & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} ct \\ x \\ y \\ z \end{pmatrix}. \tag{2.19}$$

[1] β is speed parameter. For photon $\beta = 1$.

We can see that the Lorentz boost is actually rotation in (x,ct) space by angle ω. Lorentz transformation yields two important phenomena in SR. These are *time dilation* and *length contraction*. The dilation of time measured in two inertial reference frames is described by

$$\Delta t = \Delta t_0 \gamma \tag{2.20}$$

and length contraction of matter measured in two inertial reference frames is described by

$$L = \frac{L_0}{\gamma} \tag{2.21}$$

where Δt_0 and L_0 are proper time and proper length respectively.

Chapter 3

Introduction to general relativity

3.1 Tensor and curvature

In last chapter, we consider physical phenomena based on flat space which is a spacial case of this chapter. In this chapter is curved space is of interest. A physical quantity needed here is tensors which are geometrical objects. Tensor is invariant in all coordinate systems. Vectors and scalars are subsets of tensors indicated by rank (order) of tensors i.e. vector is tensor of rank 1, scalar is zeroth-rank tensor. Any tensors are defined on **manifold** $\mathcal{M}$ which is n-dimensional generalized object that *it locally looks like Euclidian space* $\mathbb{R}^n$.

3.1.1 Transformations of scalars, vectors and tensors

In general relativity, vectors are expressed in general form, $\mathbf{X} = X^a e_a$ where X^a is a component of vector and e_a is basis vector of the component. Here $\Lambda^a{}_b$ is general transformation metric. **Contravariant vector** or **tangent vector** X^a transforms like

$$X'^a = \frac{\partial x'^a}{\partial x^b} X^b = \Lambda^a{}_b X^b \tag{3.1}$$

or can be written in chain rule,

$$\mathrm{d}x'^a = \frac{\partial x'^a}{\partial x^b}\,\mathrm{d}x^b.$$

We define **Kronecker delta**, $\delta^a{}_b$ as a quantity with values 0 or 1 under conditions,

$$\delta^a{}_b = \begin{cases} 1 & \text{if} \quad a = b \\ 0 & \text{if} \quad a \neq b. \end{cases}$$

Therefore

$$\frac{\partial x'^a}{\partial x'^b} = \frac{\partial x^a}{\partial x^b} = \delta^a{}_b. \tag{3.2}$$

Another quantity is **scalar** ϕ which is invariant under transformation,

$$\phi'(x'^a) = \phi(x^a). \tag{3.3}$$

Consider derivative of scalar field $\phi = \phi(x^a(x'))$ with respect to x'^a, using chain rule, we obtain

$$\frac{\partial \phi}{\partial x'^a} = \frac{\partial x^b}{\partial x'^a}\frac{\partial \phi}{\partial x^b}.$$

Covariant vector or **1-form** or **dual vector** X_a in the x^a-coordinate system, transforms according to

$$X'_a(x^a) = \frac{\partial x^b}{\partial x'^a} X_b(x^a) = \Lambda^b{}_a X_b(x^a). \tag{3.4}$$

There is also a relation

$$\Lambda^a{}_b \Lambda^b{}_c = \delta^a{}_c. \tag{3.5}$$

For higher rank tensor transformation follows

$$\begin{aligned} T'^{k'_1 k'_2 \ldots k'_k}{}_{l'_1 \ldots\ldots l'_l} &= \frac{\partial x'^{k'_1}}{\partial x^{k_1}} \cdots\cdots \frac{\partial x'^{k'_k}}{\partial x^{k_k}} \frac{\partial x^{l_1}}{\partial x'^{l'_1}} \cdots\cdots \frac{\partial x^{l_l}}{\partial x'^{l'_l}} T^{k_1 k_2 \ldots\ldots k_k}{}_{l_1 \ldots\ldots l_l} \\ &= \Lambda^{k'_1}{}_{k_1} \cdots\cdots \Lambda^{k'_k}{}_{k_k} \Lambda^{l_1}{}_{l'_1} \cdots\cdots \Lambda^{l_l}{}_{l'_l} T^{k_1 k_2 \ldots\ldots k_k}{}_{l_1 \ldots\ldots l_l}. \end{aligned} \tag{3.6}$$

The metric tensor

Any symmetric covariant tensor field of rank 2 such as g_{ab}, defines a metric tensor. Metrics are used to define distance and length of vectors. The square of the infinitesimal distance or interval between two neighboring points (events) is defined by

$$\mathrm{d}s^2 = g_{ab}\mathrm{d}x^a \mathrm{d}x^b. \tag{3.7}$$

Inverse of g_{ab} is g^{ab}. The metric g^{ab} is given by

$$g_{ab} g^{ac} = \delta_b{}^c. \tag{3.8}$$

We can use the metric tensor to lower and raise tensorial indices,

$$T^{\ldots\ldots}_{\ldots a \ldots} = g_{ab} T^{\ldots b \ldots}_{\ldots\ldots} \tag{3.9}$$

and

$$T^{\ldots a \ldots}_{\ldots\ldots} = g^{ab} T^{\ldots\ldots}_{\ldots b \ldots}. \tag{3.10}$$

3.1.2 Covariant derivative

Consider a contravariant vector field X^a defined along path $x^a(u)$ on manifold at a point P where u is free parameter. There is another vector $X^a(x^a+\delta x^a)$ defined at $x^a+\delta x^a$ or point Q. Therefore

$$X^a(x+\delta x) = X^a(x) + \delta X^a. \tag{3.11}$$

We are going to define a tensorial derivative by introducing a vector at Q which is parallel to X^a. We can assume the parallel vector differs from X^a by a small amount denoted by $\bar{\delta}X_a(x)$. Because covariant derivative is defined on curved geometry, therefore parallel vector are parallel only on flat geometry which is mapped from manifold. We will explain about it in next section.

The difference vector between the parallel vector and the vector at point Q illustrated in Fig. 3.1 is

$$[X^a(x)+\delta X^a(x)] - [X^a(x)+\bar{\delta}X^a(x)] = \delta X^a(x) - \bar{\delta}X^a. \tag{3.12}$$

We define the **covariant derivative** of X^μ by the limiting process

$$\begin{aligned}\nabla_c X^a &= \lim_{\delta x^c\to 0}\frac{1}{\delta x^c}\Big\{X^a(x+\delta x) - \big[X^a(x)+\bar{\delta}X^a(x)\big]\Big\}\\ &= \lim_{\delta x^c\to 0}\frac{1}{\delta x^c}\big[\delta X^a(x) - \bar{\delta}X^a\big].\end{aligned}$$

$\bar{\delta}X^a(x)$ should be a linear function on manifold. We can write

$$\bar{\delta}X^a(x) = -\Gamma^a_{bc}(x)X^b(x)\delta x^c \tag{3.13}$$

where the $\Gamma^a_{bc}(x)$ are functions of coordinates called **Cristoffel symbols** of the second kind. In flat space $\Gamma^a_{bc}(x)=0$. But in curved space it is impossible to make all the $\Gamma^a_{bc}(x)$ vanish over all space. The *Cristoffel symbols of second kind* are defined by

$$\Gamma^a_{bc} = \frac{1}{2}g^{ad}\left(\frac{\partial g_{dc}}{\partial x^b} + \frac{\partial g_{bd}}{\partial x^c} - \frac{\partial g_{cb}}{\partial x^d}\right). \tag{3.14}$$

Following from the last equation that the Cristoffel symbols are necessarily symmetric or we often called it that *torsion-free*, i.e.

$$\Gamma^a_{bc} = \Gamma^a_{cb}. \tag{3.15}$$

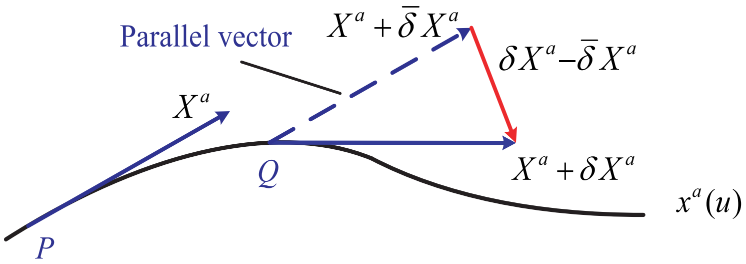

Figure 3.1: Vector along the path $x^a(u)$

We now define derivative of a vector on curved space. Using equation (3.13) and chain rule, we have

$$\nabla_c X^a \quad = \quad \lim_{\delta x^c \to 0} \frac{1}{\delta x^c}\left[\frac{\partial X^a(x)}{\partial x^c}\delta x^c + \Gamma^a_{bc}(x) X^b(x)\delta x^c\right].$$

Therefore the covariant derivative is

$$\nabla_c X^a = \partial_c X^a + \Gamma^a_{bc} X^b. \tag{3.16}$$

The notation ∂_c is introduced by $\partial_c \equiv \partial/\partial x^c$. We next define the covariant derivative of a scalar field ϕ to be the same as its ordinary derivative (prove of this identity is in Appendix A),

$$\nabla_c \phi = \partial_c \phi. \tag{3.17}$$

Consider

$$\begin{aligned}
\nabla_c \phi &= \nabla_c (X_a Y^a) \\
&= (\nabla_c X_a) Y^a + X_a (\nabla_c Y^a) \\
&= (\nabla_c X_a) Y^a + X_a (\partial_c Y^a + \Gamma^a_{bc} Y^b)
\end{aligned}$$

and

$$\partial_c \phi \quad = \quad (\partial_c X_a) Y^a + X_a (\partial_c Y^a).$$

From equation (3.17), we equate both equations together:

$$(\partial_c X_a) Y^a + X_a (\partial_c Y^a) = (\nabla_c X_a) Y^a + X_a (\partial_c Y^a + \Gamma^a_{bc} Y^b).$$

Renaming a to b and b to a for the last term in the right-hand side of the equation above because they are only dummy indices therefore,

$$(\nabla_c X_a)Y^a = (\partial_c X_a)Y^a + (\partial_c X_a - \Gamma^b_{ac}X_b)Y^a.$$

Covariant derivative for covariant vector is then

$$\nabla_c X_a = \partial_c X_a - \Gamma^b_{ac}X_b. \tag{3.18}$$

The covariant derivative for tensor follows

$$\nabla_c T^{a....}_{b....} = \partial_c T^{a....}_{b....} + \Gamma^a_{cd}T^{d....}_{b....} + ... - \Gamma^d_{cb}T^{a....}_{d....} - ..... \tag{3.19}$$

The last important formula is covariant derivative of the metric tensor,

$$\nabla_c g_{ab} = 0 \tag{3.20}$$

and

$$\nabla_c g^{ab} = 0. \tag{3.21}$$

The proves of these two identities are in Appendix A.

3.1.3 Parallel transport

The concept of parallel transport along a path is in flat space. A parallel vector transporting from a point to another point maintains its unchange in magnitude and direction. In curve space, components of a vector are expected to change under parallel transport in different way from the case of flat space. Consider the parallel transport along a curve $x^a(\tau)$ with a tangent vector $X^a = \mathrm{d}x^a/\mathrm{d}\tau$ where τ is a parameter along the curve. Beginning from the formula of the covariant derivative

$$X^a \nabla_a X^b = 0, \tag{3.22}$$

we get

$$\begin{aligned} \frac{\mathrm{d}x^a}{\mathrm{d}\tau}\left(\partial_a \frac{\mathrm{d}x^b}{\mathrm{d}\tau} + \Gamma^b_{ac}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\right) &= 0 \\ \frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\partial}{\partial x^a}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} + \Gamma^b_{ac}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} &= 0 \\ \frac{\mathrm{d}}{\mathrm{d}\tau}\left(\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\right) + \Gamma^b_{ac}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} &= 0. \end{aligned}$$

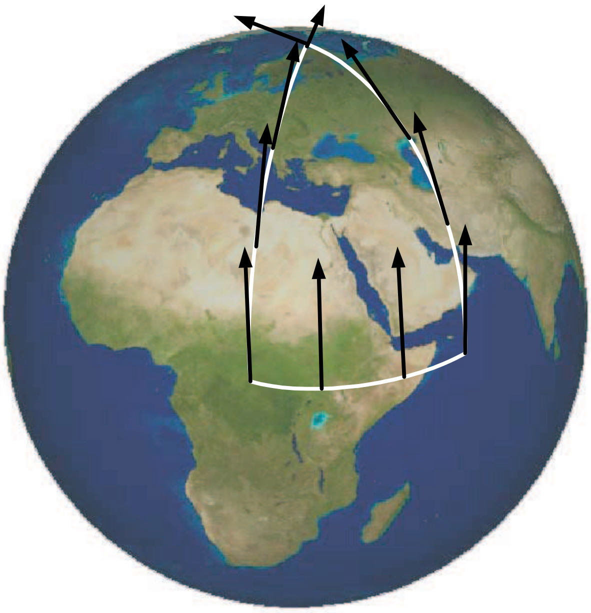

Figure 3.2: Parallel transport of vectors on curved space

We rename a to b and b to a, then

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc} \frac{\mathrm{d}x^b}{\mathrm{d}\tau} \frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0. \tag{3.23}$$

This equation is known as the **geodesic equation**. The geodesic distance between any two points is shortest. The geodesic is a curve space generalization of straight line in flat space.

3.1.4 Curvature tensor

We now discuss the most important concepts of general relativity, That is concept the of **Riemannian geometry** or curved geometry which is described in tensorial form. We will introduce the **Riemann tensor** by considering parallel transport along an infinitesimal loop illustrated in Fig. 3.3. The Riemann curvature tensor $R^a{}_{bcd}$ is defined by the commutator of covariant derivatives,

$$R^a{}_{bcd} X^b = (\nabla_c \nabla_d - \nabla_d \nabla_c) X^a. \tag{3.24}$$

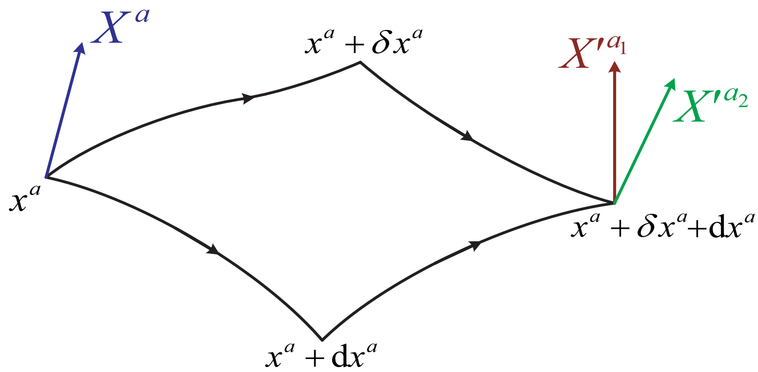

Figure 3.3: Transporting X^a around an infinitesimal loop

Consider

$$\begin{aligned}(\nabla_c\nabla_d - \nabla_d\nabla_c)X^a &= \nabla_c\nabla_d X^a - \nabla_d\nabla_c X^a \\ &= \nabla_c(\partial_d X^a + \Gamma^a_{db}X^b) - \nabla_d(\partial_c X^a + \Gamma^a_{cb}X^b).\end{aligned}$$

$\nabla_d X^a$ is a tensor type(1, 1). Using equation (3.19) we get

$$\begin{aligned}(\nabla_c\nabla_d - \nabla_d\nabla_c)X^a &= \partial_c(\partial_d X^a + \Gamma^a_{db}X^b) - \Gamma^e_{cd}(\partial_e X^a + \Gamma^a_{eb}X^b) + \Gamma^a_{ce}(\partial_d X^e + \Gamma^e_{db}X^b) \\ &\quad -\partial_d(\partial_c X^a + \Gamma^a_{cb}X^b) + \Gamma^e_{dc}(\partial_e X^a + \Gamma^a_{eb}X^b) - \Gamma^a_{de}(\partial_c X^e + \Gamma^e_{cb}X^b) \\ &= \partial_c\partial_d X^a + \Gamma^a_{db}\partial_c X^b + \partial_c\Gamma^a_{db}X^b - \Gamma^e_{cd}\partial_e X^a - \Gamma^e_{cd}\Gamma^a_{eb}X^b \\ &\quad +\Gamma^a_{ce}\partial_d X^e + \Gamma^a_{ce}\Gamma^e_{db}X^b - \partial_d\partial_c X^a - \Gamma^a_{cb}\partial_d X^b - \partial_d\Gamma^a_{cb}X^b \\ &\quad +\Gamma^e_{dc}\partial_e X^a + \Gamma^e_{dc}\Gamma^a_{eb}X^b - \Gamma^a_{de}\partial_c X^e - \Gamma^a_{de}\Gamma^e_{cb}X^b.\end{aligned}$$

We rename e to b in the terms $\Gamma^a_{ce}\partial_d X^e$ and $\Gamma^a_{de}\partial_c X^e$. Assuming torsion-free condition of Cristoffel symbols and using equation (3.24), we have Riemann tensor expressed in terms of Cristoffel symbols:

$$R^a{}_{bcd} = \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed}. \tag{3.25}$$

$R^a{}_{bcd}$ depends on the metric and the metric's first and second derivatives. It is anti-symmetric on its last pair of indices,

$$R^a{}_{bcd} = -R^a{}_{bdc}. \tag{3.26}$$

The last equation introduces identity:

$$R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0. \tag{3.27}$$

Lowering the first index with the metric, the lowered tensor is symmetric under interchanging of the first and last pair of indices. That is

$$R_{abcd} = R_{cdab}. \tag{3.28}$$

The tensor is anti-symmetric on its first pair of indices as

$$R_{abcd} = -R_{bacd}. \tag{3.29}$$

We can see that the lowered curvature tensor satisfies

$$\begin{aligned} R_{abcd} = -R_{abdc} = -R_{bacd} &= R_{cdab} \\ \text{and} \qquad R_{abcd} + R_{adbc} + R_{acdb} &= 0. \end{aligned} \tag{3.30}$$

The curvature tensor satisfies a set of differential identities called the **Bianchi identities**:

$$\nabla_a R_{bcde} + \nabla_c R_{abde} + \nabla_b R_{cade} = 0. \tag{3.31}$$

We can use the curvature tensor to define **Ricci tensor** by the contraction,

$$R_{ab} = R^c{}_{acb} = g^{cd} R_{dacb}. \tag{3.32}$$

Contraction of Ricci tensor then also defines **curvature scalar** or **Ricci scalar** R by

$$R = g^{ab} R_{ab}. \tag{3.33}$$

These two tensors can be used to define **Einstein tensor**

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R, \tag{3.34}$$

which is also symmetric. By using the equation (3.31), the Einstein tensor can be shown to satisfy the contracted Bianchi identities

$$\nabla^b G_{ab} = 0 \tag{3.35}$$

3.2 The equivalence principle

In chapter 2, we introduced some concepts about inertial frames of reference. We will discuss about its nature in this chapter.

An inertial frame is defined as one in which a free particle moves with constant velocity. However gravity is long-range force and can not be screened out. Hence an purely inertial frame is impossible to be found. We can only imagine about. According to Newtonian gravity, when gravity acts on a body, it acts on the *gravitational mass*, m_{g}. The result of the force is an acceleration of the *inertial mass*, m_{i}. When all bodies fall in vacuum with the same acceleration, the ratio of inertial mass and gravitational mass is independent of the size of bodies. Newton's theory is in principle consistent with $m_{\mathrm{i}} = m_{\mathrm{g}}$ and within high experimental accuracies $m_{\mathrm{i}} = m_{\mathrm{g}}$ to 1 in 1,000. Therefore the equivalence of gravitational and inertial mass implies:

In a small laboratory falling freely in gravitational field, mechanical phenomena are the same as those observed in an inertial frame in the absence of gravitational field.

In 1907 Einstein generalized this conclusion by replacing the word *mechanical phenomena* with *the laws of physics*. The resulting statement is known as the **principle of equivalence**. The freely-falling frames introduce the **local inertial frames** which are important in relativity.

3.3 Einstein's law of gravitation

In this section, we use Riemannian formalism to connect matter and metric that leads to a satisfied gravitational theory.

3.3.1 The energy-momentum tensor for perfect fluids

The energy-momentum tensor contains information about the total energy density measured by an arbitrary inertial observer. It is defined by the notation T^{ab}. We start by considering the simplest kind of matter field, that is **non-relativistic matter** or **dust**. We can simply construct the energy-momentum tensor for dust by using four-velocity u^a defined as $u^a = (c, 0, 0, 0)$ for rest frame and the proper density ρ:

$$T^{ab} = \rho u^a u^b. \tag{3.36}$$

Notice that T^{00} is the energy density.

In general relativity, the source of gravitational field can be regarded as **perfect fluid**. A perfect fluid in relativity is defined as a fluid that has no viscosity and no heat conduction. No heat conduction implies $T^{0i} = T^{i0} = 0$ in rest frame and energy can flow only if particles flow. No viscosity implies vanishing of force paralleled to the interface between particles. The forces should always be perpendicular to the interface, i.e. the spatial component T^{ij} should be zero unless $i = j$. As a result we write the energy-momentum tensor for perfect fluids in rest frame as

$$T^{ab} = \begin{pmatrix} \rho c^2 & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \tag{3.37}$$

where p is the pressure and ρ is the energy density. Form equation (3.37), it is easy to show that

$$T^{ab} = \Big(\rho + \frac{p}{c^2}\Big) u^a u^b + p g^{ab}. \tag{3.38}$$

We simply conclude from the above equation that the energy-momentum tensor is symmetric tensor. The metric tensor g^{ab} here is for flat spacetime (we often write η^{ab} instead of g^{ab} for flat spacetime). Notice that in the limit $p \to 0$, a perfect fluid reduces to dust equation (3.36). We can easily show that the energy-momentum tensor conserved in flat spacetime:

$$\partial_b T^{ab} = 0. \tag{3.39}$$

Moreover, if we use non-flat metric, the conservation law is

$$\nabla_b T^{ab} = 0. \tag{3.40}$$

3.3.2 Einstein's field equation

Einstein's field equation told us that the metric is correspondent to geometry and geometry is the effect of an amount of matter which is expressed in energy-momentum tensor. Matters cause spacetime curvature. We shall use Remannian formalism to connect matter and metric. Since covariant divergence of the Einstein tensor G_{ab} vanishes in equation (3.35), we therefore write

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = \kappa T_{ab}. \tag{3.41}$$

If there is gravity in regions of space there must be matter present. The proportional constant κ is arbitrary. Contract the Einstein tensor by using the metric tensor, we obtain

$$\begin{aligned} G = g^{ab} G_{ab} &= \kappa g^{ab} T_{ab} \\ g^{ab} R_{ab} - \frac{1}{2} g^{ab} g_{ab} R &= \kappa g^{ab} T_{ab} \\ R - \frac{1}{2} \delta^a_a R &= \kappa T. \end{aligned}$$

δ^a_a is 4 which is equal to its dimensions. Therefore equation (3.41) can be rewritten as

$$R_{ab} = \kappa\Big(T_{ab} - \frac{1}{2}g_{ab}T\Big). \tag{3.42}$$

We want to choose appropriate value of constant κ. Let us consider motion of particle which follows a geodesic equation

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} + \Gamma^a_{bc}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} = 0. \tag{3.43}$$

In Newtonian limit, the particles move slowly with respect to the speed of light. Four-velocity is $u^a \cong u^0$, therefore the geodesic equation reduces to

$$\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} = -\Gamma^a_{00}\frac{\mathrm{d}x^0}{\mathrm{d}\tau}\frac{\mathrm{d}x^0}{\mathrm{d}\tau} = -c^2\Gamma^a_{00}. \tag{3.44}$$

Since the field is static in time, the time derivative of the metric vanishes. Therefore the Christoffel symbol is reduced to:

$$\begin{aligned}\Gamma^a_{00} &= \frac{1}{2}g^{ad}\Big(\partial_0 g_{d0} + \partial_0 g_{0d} - \partial_d g_{00}\Big)\\ &= -\frac{1}{2}g^{ad}\partial_d g_{00}.\end{aligned} \tag{3.45}$$

In the limit of weak gravitational field, we can decompose the metric into the Minkowski metric, the equation (2.9), plus a small perturbation,

$$g_{ab} = \eta_{ab} + h_{ab}, \tag{3.46}$$

where $h_{ab} \ll 1$ here. From definition of the inverse metric, we find that to the first order in h,

$$\begin{aligned}g^{ab} &= \frac{1}{g_{ab}}\\ &= (\eta_{ab} + h_{ab})^{-1}\\ &\cong \eta^{ab} - h^{ab}.\end{aligned}$$

Since the metric here is diagonal matrix. Our approximate is to only first order for small perturbation of inverse matric. Equation (3.45) becomes

$$\begin{aligned}\Gamma^a_{00} &= \frac{1}{2}\big(\eta^{ad} - h^{ad}\big)\partial_d h_{00}\\ &\cong \frac{1}{2}\eta^{ad}\partial_d h_{00}.\end{aligned} \tag{3.47}$$

Substituting this equation into the geodesic equation (3.44),

$$\frac{\mathrm{d}^2 x^a}{\mathrm{d}\tau^2} = \frac{c^2}{2}\eta^{ad}\partial_d h_{00}. \tag{3.48}$$

Using $\partial_0 h_{00} = 0$ and since $\mathrm{d}^2x^0/\mathrm{d}\tau^2$ is zero, we are left with spacelike components of the above equation. Therefore,

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = \frac{c^2}{2}\eta^{ij}\partial_j h_{00}. \tag{3.49}$$

Recall Newton's equation of motion

$$\frac{\mathrm{d}^2 x^i}{\mathrm{d}\tau^2} = -\partial^i \Phi \tag{3.50}$$

where Φ is Newtonian potential. Comparing both equations above, we obtain

$$h_{00} = \frac{2}{c^2}\Phi. \tag{3.51}$$

In non relativistic limit (dustlike), the energy-momentum tensor reduce to the equation (3.36). We will work in fluid rest frame. Equation (3.38) gives

$$T_{00} = \rho c^2 \tag{3.52}$$

and

$$\begin{aligned} T &= g^{ab}T_{ab} \\ &\cong g^{00}T_{00} \\ &\cong \eta^{00}T_{00} \\ &= -\rho c^2. \end{aligned} \tag{3.53}$$

We insert this into the 00 component of our gravitational field equation (3.42). We get

$$R_{00} = \kappa\Big(\rho c^2 - \frac{1}{2}\rho c^2\Big) = \frac{1}{2}\kappa\rho c^2. \tag{3.54}$$

This is an equation relating derivative of the metric to the energy density. We shall expand $R^i{}_{0i0}$ in term of metric. Since $R^0{}_{000} = 0$ then

$$R_{00} = R^i{}_{0i0} = \partial_i \Gamma^i_{00} - \partial_0 \Gamma^i_{i0} + \Gamma^i_{ib}\Gamma^b_{00} - \Gamma^i_{0b}\Gamma^b_{i0}.$$

The second term is time derivative which vanishes for static field. The second order of Christoffel symbols $(\Gamma)^2$ can be neglected since we only consider a small perturbation. From this we get

$$\begin{aligned} R_{00} \cong \partial_i \Gamma^i_{00} &= \partial_i \left[\frac{1}{2} g^{ib} \Big(\partial_0 g_{b0} + \partial_0 g_{0b} - \partial_b g_{00} \Big) \right] \\ &= -\frac{1}{2} \partial_i g^{ib} \partial_b g_{00} - \frac{1}{2} g^{ib} \partial_i \partial_b g_{00} \\ &\cong -\frac{1}{2} \partial_i \eta^{ij} \partial_j h_{00} - \frac{1}{2} \eta^{ij} \partial_i \partial_j h_{00} \\ &= -\frac{1}{2} \delta^{ij} \partial_i \partial_j h_{00} \\ &= -\frac{1}{2} \nabla^2 h_{00}. \end{aligned} \tag{3.55}$$

Substituting equation (3.51) into the above equation and using equation (3.54), we finally obtain

$$\nabla^2 \Phi = -\frac{1}{2} \kappa \rho c^4. \tag{3.56}$$

The connection to Newtonian theory appears when this equation is compared with Poisson's equation for Newtonian theory of gravity. The theory of general relativity must be correspondent to Newton's non-relativistic theory in limiting case of weak field. Poisson's equation in a Newtonian gravitational field is

$$\nabla^2 \Phi = 4\pi G \rho \tag{3.57}$$

where G is classical gravitational constant. Comparing this with equation (3.56) gives the value of κ,

$$\kappa = \frac{8\pi G}{c^4}. \tag{3.58}$$

From equation (3.41) then the full field equation is

$$G_{ab} = R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi G}{c^4} T_{ab}. \tag{3.59}$$

Chapter 4

Variational principle approach to general relativity

4.1 Lagrangian formulation for field equation

4.1.1 The Einstein-Hilbert action

All fundamental physical equation of classical field including the Einstein's field equation can be derived from a variational principle. The condition required in order to get the field equation follows from

$$\delta \int \mathcal{L} \mathrm{d}^4 x = 0. \tag{4.1}$$

Of course the quantity above must be an invariant and must be constructed from the metric g_{ab} which is dynamical variable in GR. We shall not include function which is first the derivative of metric because it vanishes at a point $\mathrm{P} \in \mathcal{M}$. The Riemann tensor is of course made from second derivative set of the metrics, and the only independent scalar constructed from the metric is the Ricci scalar R. The well definition of Lagrangian density is $\mathcal{L} = \sqrt{-g} R$, therefore

$$S_{\mathrm{EH}} = \int \sqrt{-g} R \, \mathrm{d}^4 x \tag{4.2}$$

is known as the **Einstein-Hilbert action**. We derive field equation by variation of action in equation (4.2),

$$\begin{aligned}\delta S_{\rm EH} &= \delta\int\sqrt{-g}R\,{\rm d}^4x\\ &= \int{\rm d}^4x\delta\Big(\sqrt{-g}g^{ab}R_{ab}\Big)\\ &= \int{\rm d}^4x\sqrt{-g}g^{ab}\delta R_{ab}+\int{\rm d}^4x\sqrt{-g}R_{ab}\delta g^{ab}+\int{\rm d}^4xR\delta\sqrt{-g}.\end{aligned}$$

Now we have three terms of variation that

$$\delta S_{\rm EH}=\delta S_{\rm EH(1)}+\delta S_{\rm EH(2)}+\delta S_{\rm EH(3)} \tag{4.3}$$

The variation of first term is

$$\delta S_{\rm EH(1)}=\int{\rm d}^4x\sqrt{-g}g^{ab}\delta R_{ab}. \tag{4.4}$$

Considering the variation of Ricci tensor,

$$\begin{aligned}R_{ab}=R^c{}_{acb} &= \partial_c\Gamma^c_{ab}-\partial_b\Gamma^c_{ac}+\Gamma^c_{cd}\Gamma^d_{ba}-\Gamma^c_{bd}\Gamma^d_{ac} \qquad (4.5)\\ \delta R_{ab} &= \partial_c\delta\Gamma^c_{ab}-\partial_b\delta\Gamma^c_{ac}+\Gamma^d_{ba}\delta\Gamma^c_{cd}+\Gamma^c_{cd}\delta\Gamma^d_{ba}-\Gamma^d_{ac}\delta\Gamma^c_{bd}-\Gamma^c_{bd}\delta\Gamma^d_{ac}\\ &= \Big(\partial_c\delta\Gamma^c_{ab}+\Gamma^c_{cd}\delta\Gamma^d_{ba}-\Gamma^d_{ac}\delta\Gamma^c_{bd}-\Gamma^d_{bc}\delta\Gamma^c_{ad}\Big)\\ &\quad-\Big(\partial_b\delta\Gamma^c_{ac}+\Gamma^c_{bd}\delta\Gamma^d_{ac}-\Gamma^d_{ba}\delta\Gamma^c_{cd}-\Gamma^d_{bc}\delta\Gamma^c_{ad}\Big)\end{aligned}$$

and the covariant derivative formula:

$$\nabla_c\delta\Gamma^c_{ab}=\partial_c\delta\Gamma^c_{ab}+\Gamma^c_{cd}\delta\Gamma^d_{ba}-\Gamma^d_{ac}\delta\Gamma^c_{bd}-\Gamma^d_{bc}\delta\Gamma^c_{ad} \tag{4.6}$$

and also

$$\nabla_b\delta\Gamma^c_{ac}=\partial_b\delta\Gamma^c_{ac}+\Gamma^c_{bd}\delta\Gamma^d_{ac}-\Gamma^d_{ba}\delta\Gamma^c_{cd}-\Gamma^d_{bc}\delta\Gamma^c_{ad} \tag{4.7}$$

we can conclude that

$$\delta R_{ab}=\nabla_c\delta\Gamma^c_{ab}-\nabla_b\delta\Gamma^c_{ac}. \tag{4.8}$$

Therefore equation (4.4) becomes

$$\begin{aligned}\delta S_{\rm EH(1)} &= \int{\rm d}^4x\sqrt{-g}g^{ab}\Big(\nabla_c\delta\Gamma^c_{ab}-\nabla_b\delta\Gamma^c_{ac}\Big)\\ &= \int{\rm d}^4x\sqrt{-g}\Big[\nabla_c\Big(g^{ab}\delta\Gamma^c_{ab}\Big)-\delta\Gamma^c_{ab}\nabla_cg^{ab}-\nabla_b\Big(g^{ab}\nabla_b\delta\Gamma^c_{ac}\Big)+\delta\Gamma^c_{ac}\nabla_bg^{ab}\Big].\end{aligned}$$

Remembering that the covariant derivative of metric is zero. Therefore we get

$$\begin{aligned}
\delta S_{\mathrm{EH}(1)} &= \int \mathrm{d}^4x\sqrt{-g}g^{ab}\Big(\nabla_c\delta\Gamma^c_{ab} - \nabla_b\delta\Gamma^c_{ac}\Big) \\
&= \int \mathrm{d}^4x\sqrt{-g}\Big[\nabla_c\Big(g^{ab}\delta\Gamma^c_{ab}\Big) - \nabla_b\Big(g^{ab}\delta\Gamma^c_{ac}\Big)\Big] \\
&= \int \mathrm{d}^4x\sqrt{-g}\,\nabla_c\Big[g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}\Big] \\
&= \int \mathrm{d}^4x\sqrt{-g}\,\nabla_c J^c
\end{aligned}$$

where we introduce

$$J^c = g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}. \tag{4.9}$$

If J^c is a vector field over a region $\mathcal{M}$ with boundary Σ. Stokes's theorem for the vector field is

$$\int_{\mathcal{M}} \mathrm{d}^4x\sqrt{|g|}\,\nabla_c J^c = \int_{\Sigma} \mathrm{d}^3x\sqrt{|h|}n_c J^c \tag{4.10}$$

where n_c is normal unit vector on hypersurface Σ. The normal unit vector n_c can be normalized by $n_a n^a = -1$. The tensor h_{ab} is induced metric associated with hypersurface defined by

$$h_{ab} = g_{ab} + n_a n_b. \tag{4.11}$$

Therefore the first term of action becomes

$$\delta S_{\mathrm{EH}(1)} = \int_{\Sigma} \mathrm{d}^3x\sqrt{|h|}n_c J^c = 0. \tag{4.12}$$

This equation is an integral with respect to the volume element of the covariant divergence of a vector. Using Stokes's theorem, this is equal to a boundary contribution at infinity which can be set to zero by vanishing of variation at infinity. Therefore this term contributes nothing to the total variation.

4.1.2 Variation of the metrics

Firstly we consider metric g_{ab}. Since the contravariant and covariant metrics are symmetric matrices then,

$$g_{ca}g^{ab} = \delta_c{}^b. \tag{4.13}$$

We now consider inverse of the metric:

$$g^{ab} = \frac{1}{g}(A^{ab})^{\mathrm{T}} = \frac{1}{g}A^{ba} \tag{4.14}$$

where g is determinant and A^{ba} is the cofactor of the metric g_{ab}. Let us fix a, and expand the determinant g by the ath row. Then

$$g = g_{ab}A^{ab}. \tag{4.15}$$

If we perform partial differentiation on both sides with respect to g_{ab}, then

$$\frac{\partial g}{\partial g_{ab}} = A^{ab}. \tag{4.16}$$

Let us consider variation of determinant g:

$$\begin{aligned} \delta g &= \frac{\partial g}{\partial g_{ab}}\delta g_{ab} \\ &= A^{ab}\delta g_{ab} \\ &= gg^{ba}\delta g_{ab}. \end{aligned}$$

Remembering that g^{ab} is symmetric, we get

$$\delta g = gg^{ab}\delta g_{ab}. \tag{4.17}$$

Using relation obtained above, we get

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2\sqrt{-g}}\,\delta g \\ &= \frac{1}{2}\frac{g}{\sqrt{-g}}\,g^{ab}\delta g_{ab}. \end{aligned} \tag{4.18}$$

We shall convert from δg_{ab} to δg^{ab} by considering

$$\begin{aligned} \delta\delta_a{}^d = \delta(g_{ac}g^{cd}) &= 0 \\ g^{cd}\delta g_{ac} + g_{ac}\delta g^{cd} &= 0 \\ g^{cd}\delta g_{ac} &= -g_{ac}\delta g^{cd}. \end{aligned}$$

Multiply both side of this equation by g_{bd} we therefore have

$$\begin{aligned} g_{bd}g^{cd}\delta g_{ac} &= -g_{bd}g_{ac}\delta g^{cd} \\ \delta_b^c\delta g_{ac} &= -g_{bd}g_{ac}\delta g^{cd} \\ \delta g_{ab} &= -g_{ac}g_{bd}\delta g^{dc}. \end{aligned} \tag{4.19}$$

Substituting this equation in to equation (4.18) we obtain

$$\begin{aligned} \delta\sqrt{-g} &= -\frac{1}{2}\sqrt{-g}\,g^{ab}g_{ac}g_{bd}\delta g^{dc} \\ &= -\frac{1}{2}\sqrt{-g}\,\delta_c^b g_{bd}\delta g^{dc} \\ &= -\frac{1}{2}\sqrt{-g}\,g_{cd}\delta g^{dc}. \end{aligned}$$

Renaming indices c to a and d to b, we get

$$\delta\sqrt{-g} \;=\; -\frac{1}{2}\sqrt{-g}\,g_{ab}\delta g^{ab}. \tag{4.20}$$

The variation of Einstein-Hilbert action becomes

$$\begin{aligned}\delta S_{\mathrm{EH}} &= \int \mathrm{d}^4x\sqrt{-g}R_{ab}\delta g^{ab} - \frac{1}{2}\int \mathrm{d}^4x R\sqrt{-g}\,g_{ab}\delta g^{ab} \\ &= \int \mathrm{d}^4x\sqrt{-g}\Big[R_{ab} - \frac{1}{2}g_{ab}R\Big]\delta g^{ab} \end{aligned}\tag{4.21}$$

The functional derivative of the action satisfies

$$\delta S = \int \sum_i \Big(\frac{\delta S}{\delta \Phi^i}\delta\Phi^i\Big)\mathrm{d}^4x \tag{4.22}$$

where $\{\Phi^i\}$ is a complete set of field varied. Stationary points are those for which $\delta S/\delta\Phi^i = 0$. We now obtain Einstein's equation in vacuum:

$$\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{EH}}}{\delta g^{ab}} = R_{ab} - \frac{1}{2}g_{ab}R = 0. \tag{4.23}$$

4.1.3 The full field equations

Previously we derived Einstein's field equation in *vacuum* due to including only gravitational part of the action but not matter field part. To obtain the full field equations, we assume that there is other field presented beside the gravitational field. The action is then

$$S = \frac{1}{16\pi G}S_{\mathrm{EH}} + S_{\mathrm{M}} \tag{4.24}$$

where S_{M} is the action for matter. We normalize the gravitational action so that we get the right answer. Following the above equation we have

$$\frac{1}{\sqrt{-g}}\frac{\delta S}{\delta g^{ab}} = \frac{1}{16\pi G}\Big(R_{ab} - \frac{1}{2}g_{ab}R\Big) + \frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{ab}} = 0.$$

We now define the energy-momentum tensor as

$$T_{ab} = -2\frac{1}{\sqrt{-g}}\frac{\delta S_{\mathrm{M}}}{\delta g^{ab}}. \tag{4.25}$$

This allows us to recover the complete Einstein's equation,

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi G T_{ab}. \tag{4.26}$$

4.2 Geodesic equation from variational principle

Consider motion of particle along a path $x^a(\tau)$. We will perform a variation on this path between two points P and Q. The action is simply

$$S = \int \mathrm{d}\tau. \tag{4.27}$$

In order to perform the variation, it is useful to introduce an arbitrary auxiliary parameter s. Here ds is displacement on spacetime. We have

$$\mathrm{d}\tau = \Big(- g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big)^{1/2}\mathrm{d}s \tag{4.28}$$

We vary the path by using standard procedure:

$$\begin{aligned}
\delta S &= \delta \int \mathrm{d}\tau \\
&= \int \delta\Big(- g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big)^{1/2}\mathrm{d}s \\
&= \frac{1}{2}\int \mathrm{d}s\Big(- g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big)^{-1/2}\Big[- \delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} - 2g_{ab}\frac{\mathrm{d}\delta x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big].
\end{aligned}$$

Considering the last term,

$$-2g_{ab}\frac{\mathrm{d}\delta x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\Big[- 2g_{ab}\delta x^a\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big] + 2\frac{\mathrm{d}g_{ab}}{\mathrm{d}s}\delta x^a\frac{\mathrm{d}x^b}{\mathrm{d}s} + 2g_{ab}\delta x^a\frac{\mathrm{d}^2x^b}{\mathrm{d}s^2}.$$

The two points, P and Q are fixed. We can set first term of above equation to zero. Therefore we obtain

$$\begin{aligned}
\delta S &= \frac{1}{2}\int \mathrm{d}s\Big(- g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\Big)^{-1/2}\Big[- \delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} + 2\frac{\mathrm{d}g_{ab}}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\delta x^a + 2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}s^2}\delta x^a\Big] \\
&= \frac{1}{2}\int \mathrm{d}\tau\frac{\mathrm{d}s^2}{\mathrm{d}\tau^2}\Big[- \delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s} + 2\frac{\mathrm{d}g_{ab}}{\mathrm{d}s}\frac{\mathrm{d}x^b}{\mathrm{d}s}\delta x^a + 2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}s^2}\delta x^a\Big] \\
&= \frac{1}{2}\int \mathrm{d}\tau\Big[- \delta g_{ab}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau} + 2\frac{\mathrm{d}g_{ab}}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^a + 2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}\tau^2}\delta x^a\Big].
\end{aligned}$$

By using chain rule we get

$$\begin{aligned}
\delta S &= \frac{1}{2}\int \mathrm{d}\tau\Big[- \frac{\partial g_{ab}}{\partial x^c}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^c + 2\frac{\partial g_{ab}}{\partial x^c}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^b}{\mathrm{d}\tau}\delta x^a + 2g_{ab}\frac{\mathrm{d}^2x^b}{\mathrm{d}\tau^2}\delta x^a\Big] \\
&= \frac{1}{2}\int \mathrm{d}\tau\Big[- \partial_b g_{ac}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\delta x^b + \partial_c g_{ba}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\delta x^b + \partial_a g_{bc}\frac{\mathrm{d}x^c}{\mathrm{d}\tau}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\delta x^b + 2g_{ba}\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2}\delta x^b\Big] \\
&= \int \mathrm{d}\tau\delta x^b\Big[\frac{1}{2}\Big(- \partial_b g_{ac} + \partial_c g_{ba} + \partial_a g_{bc}\Big)\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} + g_{ba}\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2}\Big].
\end{aligned}$$

We set the variation of action to zero, $\delta S = 0$ and multiply by g^{db},

$$\frac{1}{2}g^{db}\Big(- \partial_b g_{ac} + \partial_c g_{ba} + \partial_a g_{bc}\Big)\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} + \delta^d_a\frac{\mathrm{d}^2x^a}{\mathrm{d}\tau^2} = 0$$

We therefore recover geodesic equation:

$$\frac{\mathrm{d}^2x^d}{\mathrm{d}\tau^2} + \Gamma^d_{ac}\frac{\mathrm{d}x^a}{\mathrm{d}\tau}\frac{\mathrm{d}x^c}{\mathrm{d}\tau} \quad = \quad 0 \tag{4.29}$$

4.3 Field equation with surface term

In section 4.1 we have derived field equation without boundary term which is set to be zero at infinity. In this section, we shall generalize field equation to general case which includes boundary term in action.

4.3.1 The Gibbons-Hawking boundary term

We begin by putting boundary term in the first part of the action in section 4.1 of Einstein-Hilbert action,

$$\delta S_{\mathrm{EH}} = \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}\Big(R_{ab} - \frac{1}{2}g_{ab}R\Big)\delta g^{ab}\mathrm{d}^4x + \frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}n_c J^c \mathrm{d}^3x. \tag{4.30}$$

Considering vector J^c in the last term of equation (4.9) we have

$$J^c = g^{ab}\delta\Gamma^c_{ab} - g^{ac}\delta\Gamma^b_{ab}.$$

Using formula (prove in appendix B)

$$\delta\Gamma^c_{ab} = \frac{1}{2}g^{cd}\Big(\nabla_a\delta g_{bd} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}\Big), \tag{4.31}$$

we get

$$\begin{aligned} J^c \quad &= \quad g^{ab}\Big[\frac{1}{2}g^{cd}\Big(\nabla_a\delta g_{bd} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}\Big)\Big] \\ & \quad\quad -g^{ac}\Big[\frac{1}{2}g^{bd}\Big(\nabla_a\delta g_{bd} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}\Big)\Big] \\ &= \quad \frac{1}{2}g^{ab}g^{cd}\nabla_a\delta g_{bd} + \frac{1}{2}g^{ab}g^{cd}\nabla_b\delta g_{ad} - \frac{1}{2}g^{ab}g^{cd}\nabla_d\delta g_{ab} \\ & \quad\quad -\frac{1}{2}g^{ac}g^{bd}\nabla_a\delta g_{bd} - \frac{1}{2}g^{ac}g^{bd}\nabla_b\delta g_{ad} + \frac{1}{2}g^{ac}g^{bd}\nabla_d\delta g_{ab}. \end{aligned}$$

Interchanging dummy indices a and b in the second term, a and d in the fourth term, and a and d in the last term, we obtain result

$$J^c = g^{ab} g^{cd} \big(\nabla_a \delta g_{bd} - \nabla_d \delta g_{ab} \big). \tag{4.32}$$

Now our discussion is on hypersurface. Lowering index of J^c with metric g_{ce},

$$J_c = g^{ab} \big(\nabla_a \delta g_{bd} - \nabla_d \delta g_{ab} \big). \tag{4.33}$$

By using equation (4.11), we have

$$\begin{aligned} n^c J_c &= n^c \big(h^{ab} - n^a n^b \big) \big(\nabla_a \delta g_{bd} - \nabla_d \delta g_{ab} \big) \\ &= n^c h^{ab} \nabla_a \delta g_{bd} - n^c h^{ab} \nabla_d \delta g_{ab} - n^a n^b n^c \nabla_a \delta g_{bd} + n^a n^b n^c \nabla_d \delta g_{ab}. \end{aligned}$$

From boundary condition, the first term of this equation vanishes since it is projected on hypersurface which variation of metric and induced metric vanish at Σ such as

$$\delta g^{ab} = 0, \tag{4.34}$$

$$\delta h^{ab} = 0 \tag{4.35}$$

and we use definition

$$n^a n^b n^c = 0. \tag{4.36}$$

Therefore we obtain

$$n^c J_c = -n^c h^{ab} \nabla_d \delta g_{ab}. \tag{4.37}$$

Now let us consider arbitrary tensor $T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l}$ at P $\in \Sigma$. It is a tensor on the tangent space to Σ at P if

$$T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = h^{a_1}{}_{c_1} \ldots h^{a_k}{}_{c_k} h^{d_1}{}_{b_1} \ldots h^{d_l}{}_{b_l} T^{c_1 \ldots b_k}{}_{d_1 \ldots d_l} \tag{4.38}$$

Defining $h_a{}^b \nabla_b$ as a projected covariant derivative on hypersurface by using notation D_a, it satisfies

$$D_c T^{a_1 \ldots a_k}{}_{b_1 \ldots b_l} = h^{a_1}{}_{d_1} \ldots h^{a_k}{}_{d_k} h^{e_1}{}_{b_1} \ldots h^{e_l}{}_{b_l} h^f{}_c \nabla_f T^{d_1 \ldots d_k}{}_{e_1 \ldots e_k}. \tag{4.39}$$

The covariant derivative for induced metric on hypersurface automatically satisfies

$$D_a h_{bc} = h_a{}^d h_b{}^e h_c{}^f \nabla_d \big(g_{af} + n_e n_f \big) = 0 \tag{4.40}$$

since $\nabla_d g_{ef} = 0$ and $h_{ab} n^b = 0$ (remembering n^b is perpendicular to hypersurface. Therefore its dot product with metric on hypersurface is zero). Next we introduce extrinsic curvature in the form

$$K_{ab} = h_a{}^c \nabla_c n_b \tag{4.41}$$

and we can use contract it with induced metric,

$$K = h^{ab}K_{ab} = h^{ab}h_a{}^c\nabla_c n_b = h^{ab}\nabla_a n_b. \tag{4.42}$$

Variation of extrinsic curvature given by

$$\begin{aligned}
\delta K &= \delta\big(h^{ab}\nabla_a n_b\big) \\
&= \delta h^{ab}\nabla_a n_b + h^{ab}\partial_a\delta n_b - h^{ab}\delta\Gamma^c_{ab}n_c - h^{ab}\Gamma^c_{ab}\delta n_c \\
&= h^{ab}\partial_a\delta n_b - h^{ab}\Gamma^c_{ab}\delta n_c - h^{ab}\delta\Gamma^c_{ab}n_c \\
&= h^{ab}\nabla_a\delta n_b - h^{ab}\delta\Gamma^c_{ab}n_c \\
&= h^{bc}h^a{}_c\nabla_a\delta n_b - h^{ab}\delta\Gamma^c_{ab}n_c \\
&= D_c\big(h^{bc}\delta n_b\big) - h^{ab}\delta\Gamma^c_{ab}n_c \\
&= D_c\Big[\delta\big(h^{bc}n_b\big) - n_b\delta h^{bc}\Big] - h^{ab}\delta\Gamma^c_{ab}n_c \\
&= -h^{ab}\delta\Gamma^c_{ab}n_c.
\end{aligned} \tag{4.43}$$

Using equation (4.31),

$$\begin{aligned}
\delta K &= -h^{ab}n_c\Big[\frac{1}{2}g^{cd}\big(\nabla_a\delta g_{bd} + \nabla_b\delta g_{ad} - \nabla_d\delta g_{ab}\big)\Big] \\
&= \frac{1}{2}h^{ab}n^d\nabla_d\delta g_{ab}.
\end{aligned} \tag{4.44}$$

Notice that when substituting equation (4.37) to this equation, we obtain

$$\delta K = -\frac{1}{2}n^c J_c. \tag{4.45}$$

Considering boundary term from Einstein-Hilbert action

$$\begin{aligned}
\int_\Sigma \sqrt{-h}n_c J^c \mathrm{d}^3x &= \int_\Sigma \sqrt{-h}n^c J_c \mathrm{d}^3x \\
&= -2\int_\Sigma \sqrt{-h}\delta K \mathrm{d}^3x \\
&= -2\delta\int_\Sigma \sqrt{-h}K\mathrm{d}^3x + 2\int_\Sigma \delta\sqrt{-h}K\mathrm{d}^3x
\end{aligned}$$

and using equation (4.20),

$$\int_\Sigma \sqrt{-h}n_c J^c\mathrm{d}^3x = -2\delta\int_\Sigma\sqrt{-h}K\mathrm{d}^3x - \int_\Sigma\sqrt{-h}Kh_{ab}\delta h^{ab}\mathrm{d}^3x.$$

Since δh^{ab} vanishes at boundary. Therefore we get

$$\int_\Sigma \sqrt{-h}n_c J^c\mathrm{d}^3x = -2\delta\int_\Sigma\sqrt{-h}K\mathrm{d}^3x. \tag{4.46}$$

Now we have variation of Einstein-Hilbert action in the form

$$\delta S_{\rm EH} = \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}\Big(R_{ab}-\frac{1}{2}g_{ab}R\Big)\delta g^{ab}{\rm d}^4x - \frac{1}{8\pi G}\delta\int_{\Sigma}\sqrt{-h}K{\rm d}^3x. \qquad (4.47)$$

We generalize this equation by naming the first term as the variation of gravitational action, $\delta S_{\rm Gravity}$, we finally get

$$\begin{aligned}
\delta S_{\rm EH} &= \delta S_{\rm Gravity} - \frac{1}{8\pi G}\delta\int_{\Sigma}\sqrt{-h}K{\rm d}^3x \\
\delta S_{\rm Gravity} &= \delta S_{\rm EH} + \frac{1}{8\pi G}\delta\int_{\Sigma}\sqrt{-h}K{\rm d}^3x \\
\therefore \qquad S_{\rm Gravity} &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}R{\rm d}^4x + \frac{1}{8\pi G}\int_{\Sigma}\sqrt{-h}K{\rm d}^3x. \qquad (4.48)
\end{aligned}$$

The last term is known as **Gibbons-Hawking boundary term**. In next section, we will vary this term using junction condition.

4.3.2 Israel junction condition

We begin from considering the action

$$\begin{aligned}
S_{\rm Gravity} &= S_{\rm EH} + S_{\rm GH} \\
&= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}R{\rm d}^4x + \frac{1}{8\pi G}\int_{\Sigma}\sqrt{-h}K{\rm d}^3x.
\end{aligned}$$

Its variation is

$$\begin{aligned}
\delta S_{\rm Gravity} &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}G_{ab}\delta g^{ab}{\rm d}^4x + \frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}n_aJ^a{\rm d}^3x \\
&\quad + \frac{1}{8\pi G}\int_{\Sigma}\delta\sqrt{-h}K{\rm d}^3x + \frac{1}{8\pi G}\int_{\Sigma}\sqrt{-h}\delta K{\rm d}^3x. \qquad (4.49)
\end{aligned}$$

In this section, we are not interested in boundary when performing variation of gravitational action then there is no boundary condition. Variation of K is

$$\begin{aligned}
\delta K &= \delta\big({h_b}^a\nabla_a n^b\big) \\
&= \delta {h_b}^a\nabla_a n^b + {h_b}^a\partial_a\delta n^b - {h_b}^a\delta\Gamma^b_{ac}n^c + {h_b}^a\Gamma^b_{ac}\delta n^c \\
&= \delta {h_b}^a\nabla_a n^b + {h_b}^a\delta\Gamma^b_{ac}n^c + {h_b}^a\nabla_a\delta n^b. \qquad (4.50)
\end{aligned}$$

Considering the first term of equation (4.50),

$$\begin{aligned}
\delta {h_b}^a\nabla_a n^b &= \delta\big({\delta_b}^a + n^a n_b\big)\nabla_a n^b \\
&= \big(n^a\delta n_b + n_b\delta n^a\big)\nabla_a n^b.
\end{aligned}$$

Using identity

$$\delta n_b = n_b n_c \delta n^c, \tag{4.51}$$

we get

$$\begin{aligned}
\delta h_b{}^a \nabla_a n^b &= \big(n^a n_b n_c \delta n^c + n_b \delta n^a\big) \nabla_a n^b \\
&= \big(n^a n_c \delta n^c + \delta_c{}^a \delta n^c\big) n_b \nabla_a n^b \\
&= n_b \delta n^c h_c{}^a \nabla_a n^b \\
&= n_b \delta n^c K_c{}^b \\
&= 0.
\end{aligned} \tag{4.52}$$

Since extrinsic curvature is associated with hypersurface then its dot product with normal vector on Σ vanishes yielding equation (4.52). We continue to do variation of the second term of the equation (4.50) by using formula (4.31), we then have

$$h_b{}^a \delta\Gamma^b_{ac} n^c = \frac{1}{2} h_b{}^a n^c g^{bd} \Big(\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}\Big).$$

Therefore variation of K is

$$\delta K = \frac{1}{2} h^{ad} n^c \Big(\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}\Big) + h_b{}^a \nabla_a \delta n^b. \tag{4.53}$$

Applying similar procedure to the equation (4.20) for $\sqrt{-h}$, we obtain

$$\delta\sqrt{-h} = -\frac{1}{2}\sqrt{-h}\, h_{ab} \delta h^{ab}. \tag{4.54}$$

Using the equation (4.32), finally the gravitational action is

$$\begin{aligned}
\delta S_{\text{Gravity}} &= \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g} G_{ab} \delta g^{ab} \mathrm{d}^4 x - \frac{1}{16\pi G}\int_\Sigma \sqrt{-h}\, h_{ab} K \delta h^{ab} \mathrm{d}^3 x \\
&+ \frac{1}{16\pi G}\int_\Sigma \sqrt{-h} n_a g^{cb} g^{ad} \big(\nabla_c \delta g_{bd} - \nabla_d \delta g_{cb}\big) \mathrm{d}^3 x \\
&+ \frac{1}{16\pi G}\int_\Sigma \sqrt{-h} \Big[h^{ad} n^c \Big(\nabla_a \delta g_{cd} + \nabla_c \delta g_{ad} - \nabla_d \delta g_{ac}\Big) + 2 h_b{}^a \nabla_a \delta n^b\Big] \mathrm{d}^3 x.
\end{aligned}$$

Considering

$$n_a g^{cb} g^{ad} = n^d \big(h^{cb} + n^c n^b\big) = n^d h^{cb}. \tag{4.55}$$

Therefore

$$\begin{aligned}
\delta S_{\text{Gravity}} &= \frac{1}{16\pi G}\int_{\mathcal{M}} \sqrt{-g} G_{ab} \delta g^{ab} \mathrm{d}^4 x - \frac{1}{16\pi G}\int_\Sigma \sqrt{-h}\, h_{ab} K \delta h^{ab} \mathrm{d}^3 x \\
&+ \frac{1}{16\pi G}\int_\Sigma \sqrt{-h} \Big[\Big(n^d h^{cb} \nabla_c \delta g_{bd} - n^d h^{cb} \nabla_d \delta g_{cb} + h^{ad} n^c \nabla_a \delta g_{cd} \\
&+ h^{ad} n^c \nabla_c \delta g_{ad} - h^{ad} n^c \nabla_d \delta g_{ac}\Big) + 2 h_b{}^a \nabla_a \delta n^b\Big] \mathrm{d}^3 x.
\end{aligned}$$

Interchanging dummy indices between d and c and rename b to a of the first and the second term in bracket of the third integral term results that the first term and the last term in bracket cancel out. So do the second and the forth term in bracket of the third integral term. Therefore

$$\begin{aligned} \delta S_{\text{Gravity}} &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}G_{ab}\delta g^{ab}\mathrm{d}^4x - \frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}\,h_{ab}K\delta h^{ab}\mathrm{d}^3x \\ &\quad +\frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}\Big(h^{ad}n^c\nabla_a\delta g_{cd} + 2h_b{}^a\nabla_a\delta n^b\Big)\mathrm{d}^3x. \end{aligned} \tag{4.56}$$

Considering the first term in the third integral term

$$h^{ad}n^c\nabla_a\delta g_{cd} = h^{ad}\nabla_a(n^c\delta g_{cd}) - h^{ad}\delta g_{cd}\nabla_a n^c.$$

Using equation (4.19), we get

$$\begin{aligned} h^{ad}n^c\nabla_a\delta g_{cd} &= h^{ad}\nabla_a(n^c\delta g_{cd}) + h^{ad}g_{de}g_{cf}\delta g^{ef}\nabla_a n^c \\ &= h^{ad}\nabla_a(n^c\delta g_{cd}) + h^{ad}(h_{de}+n_dn_e)g_{cf}\delta g^{ef}\nabla_a n^c \\ &= h^{ad}\nabla_a(n^c\delta g_{cd}) + h^a{}_e\delta g^{ef}\nabla_a n_f \\ &= h^{ad}\nabla_a(n^c\delta g_{cd}) + K_{ef}\delta g^{ef}. \end{aligned}$$

Considering

$$\begin{aligned} \delta g^{ef} &= \delta(h^{ef}+n^en^f) \\ &= \delta h^{ef} + n^f\delta n^e + n^e\delta n^f \\ &= \delta h^{ef} + n^fn^en^a\delta n_a + n^en^fn^b\delta n_b \\ &= \delta h^{ef}. \end{aligned} \tag{4.57}$$

Therefore

$$h^{ad}n^c\nabla_a\delta g_{cd} = h^{ad}\nabla_a(n^c\delta g_{cd}) + K_{ef}\delta h^{ef}. \tag{4.58}$$

Substituting this equation in equation (4.56), we obtain

$$\begin{aligned} \delta S_{\text{Gravity}} &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}G_{ab}\delta g^{ab}\mathrm{d}^4x - \frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}\,h_{ab}K\delta h^{ab}\mathrm{d}^3x \\ &\quad +\frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}\Big[h^{ad}\nabla_a\big(n^c\delta g_{cd}\big) + K_{ef}\delta h^{ef} + 2h_b{}^a\nabla_a\delta n^b\Big]\mathrm{d}^3x \\ &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}G_{ab}\delta g^{ab}\mathrm{d}^4x + \frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}\big(K_{ab}-h_{ab}K\big)\delta h^{ab}\mathrm{d}^3x \\ &\quad +\frac{1}{16\pi G}\int_{\Sigma}\sqrt{-h}\Big[h^{ad}\nabla_a\big(n^c\delta g_{cd}\big) + 2h_b{}^a\nabla_a\delta n^b\Big]\mathrm{d}^3x \end{aligned} \tag{4.59}$$

Considering the two terms in bracket of the last integral term

$$\begin{aligned} h^{ad}\nabla_a(n^c\delta g_{cd})+2h_b{}^a\nabla_a\delta n^b &= h^{ad}\nabla_a\Big[\delta\big(n^c g_{cd}\big)-g_{cd}\delta n^c\Big]+2h_b{}^a\nabla_a\delta n^b \\ &= h^{ad}\nabla_a\delta n_d-h^a{}_c\nabla_a\delta n^c+2h_b{}^a\nabla_a\delta n^b \\ &= h^{ad}\nabla_a\delta n_d+h_b{}^a\nabla_a\delta n^b \\ &= g^{bd}h_b{}^a\nabla_a\delta n_d+h_b{}^a\nabla_a\delta n^b \\ &= h_b{}^a\nabla_a\big(g^{bd}\delta n_d+\delta n^b\big) \\ &= h_b{}^a\nabla_a\big(g^{bd}n_d n_c\delta n^c+\delta^b{}_c\delta n^c\big) \\ &= h_b{}^a\nabla_a\Big[\big(n^b n_c+\delta^b{}_c\big)\delta n^c\Big] \\ &= D_b\big(h^b{}_c\delta n^c\big) \end{aligned} \tag{4.60}$$

and substituting the last equation into equation (4.59), we obtain

$$\begin{aligned} \delta S_{\text{Gravity}} &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}G_{ab}\delta g^{ab}\mathrm{d}^4x+\frac{1}{16\pi G}\int_\Sigma\sqrt{-h}\big(K_{ab}-h_{ab}K\big)\delta h^{ab}\mathrm{d}^3x \\ &\quad+\frac{1}{16\pi G}\int_\Sigma\sqrt{-h}D_b\big(h^b{}_c\delta n^c\big)\mathrm{d}^3x. \end{aligned} \tag{4.61}$$

Since the last term in the equation (4.61) is divergence term. It yields vanishing of this integral with boundary at infinity on Σ. Finally, the variation of gravity is

$$\begin{aligned} \delta S_{\text{Gravity}} &= \frac{1}{16\pi G}\int_{\mathcal{M}}\sqrt{-g}G_{ab}\delta g^{ab}\mathrm{d}^4x \\ &\quad+\frac{1}{16\pi G}\int_\Sigma\sqrt{-h}\big(K_{ab}-h_{ab}K\big)\delta h^{ab}\mathrm{d}^3x. \end{aligned} \tag{4.62}$$

The action for matter on hypersurface is

$$S_{\text{mat}}=\int_\Sigma\mathcal{L}_{\text{mat}}\mathrm{d}^3x \tag{4.63}$$

and its variation is

$$\delta S_{\text{mat}}=\int_\Sigma\sqrt{-h}t_{ab}\delta h^{ab} \tag{4.64}$$

where the energy-momentum tensor on hypersurface is

$$t_{ab}=2\frac{1}{\sqrt{-h}}\frac{\delta S_{\text{mat}}}{\delta h^{ab}}. \tag{4.65}$$

The total action is

$$S_{\text{Gravity}}=S_{\text{EH}}+S_{\text{GB}}+S_{\text{mat}}. \tag{4.66}$$

Since we include energy-momentum tensor on hypersurface, then Einstein tensor in the bulk vanishes.[1] Therefore the total action gives the **Israel junction condition**

$$K_{ab} - h_{ab}K = -8\pi G t_{ab}. \tag{4.67}$$

The energy-momentum tensor on hypersurface is not necessarily conserved because energy can flow from the hypersurface to the bulk. The idea has been recently widely applied to the research field of braneworld gravity and braneworld cosmology (see e.g. [8] for review of application of Israel junction condition to braneworld gravity.)

[1]The bulk is the region of one dimension left beyond the hypersurface. The bulk and hypersurface together form total region of the manifold.

Chapter 5

Conclusion

Relativity begins with concept of inertial reference frame which is defined by Newton's first law. Therefore relative frames in one direction is considered allowing us to do Galilean transformation for inertial frame moving not so fast. However under this transformation, the speed of light is no longer invariant and electromagnetic wave equations are neither invariant. Special relativity came along based on Einstein's principle of spacial relativity. In this theory, there is a unification of time and space so called spacetime. It uses Lorentz transformation under which electromagnetic wave equations are invariant. Moreover it reduces to Galilean transformation in case of small velocity. Therefore SR implies no absolute inertial reference frame in the universe. Einstein introduced transformation law when including effects of gravitational field and also introduced the equivalence principle which suggest that purely inertial reference frame is not sensible. It introduces us local inertial frame in gravitational field. Therefore Newtonian mechanics based on inertial frame fails to explain gravity. SR based on flat space is extended to general relativity using concept of curved space. Curved space is indicated by Riemann tensor, $R^a{}_{bcd}$. This theory attempts to explain gravity with geometry. It tells us that mass is in fact curvature of geometry This fact is expressed in the Einstein's field equation,

$$G_{ab} = 8\pi G T_{ab}.$$

All physical laws are believed to obey principle of least action, $\delta S = 0$. The dynamical variable in GR is g_{ab} therefore the Lagrangian density in action must be a function of g_{ab}. We used Ricci scalar in term of second derivative of metric. We did not choose function of first derivative of metric since it vanishes at any point on manifold. With the action

$$S_{\mathrm{EH}} = \int \sqrt{-g} R \mathrm{d}^4 x$$

we can derive Einstein's field equation by neglecting boundary term (surface term). If including boundary term then, we must have dynamical variable on boundary or hypersurface, Σ such as h_{ab} which is an induced metric on the hypersurface. Bounding the manifold, we obtain extrinsic curvature K which is constructed from h_{ab} in the action on the hypersurface. That is

$$S_{\mathrm{GH}} = \int_{\Sigma} \sqrt{-h} K \mathrm{d}^3 x.$$

This is called Gibbon-Howking boundary term. The result of this variation is Israel junction condition,

$$K_{ab} - h_{ab} K = -8\pi G t_{ab}$$

which is shown in detail calculations in this report. This result can be applied widely to braneworld gravity.

Appendix A

Proofs of identities

A.1 $\nabla_c g_{ab} = 0$

The identity $\nabla_c g_{ab} = 0$ is proved by using covariant derivative to tensor type(0, 2):

$$\nabla_c g_{ab} = \partial_c g_{ab} - \Gamma^d_{ca} g_{bd} - \Gamma^d_{cb} g_{ad} \tag{A.1}$$

Using equation (3.14), we have

$$\begin{aligned}
\nabla_c g_{ab} &= \partial_c g_{ab} - \frac{1}{2} g_{bd} g^{de}\Big(\partial_c g_{ae} + \partial_a g_{ce} - \partial_e g_{ac}\Big) - \frac{1}{2} g_{ad} g^{de}\Big(\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{bc}\Big) \\
&= \partial_c g_{ab} - \frac{1}{2}\delta^b_e\Big(\partial_c g_{ae} + \partial_a g_{ce} - \partial_e g_{ac}\Big) - \frac{1}{2}\delta^a_e\Big(\partial_c g_{be} + \partial_b g_{ce} - \partial_e g_{bc}\Big) \\
&= \partial_c g_{ab} - \frac{1}{2}\Big(\partial_c g_{ab} + \partial_a g_{cb} - \partial_b g_{ac}\Big) - \frac{1}{2}\Big(\partial_c g_{ba} + \partial_b g_{ca} - \partial_a g_{bc}\Big) \\
&= 0.
\end{aligned} \tag{A.2}$$

A.2 $\nabla_c g^{ab} = 0$

The identity $\nabla_c g^{ab} = 0$ is proved by using covariant derivative to tensor type(2, 0):

$$\nabla_c g^{ab} = \partial_c g^{ab} + \Gamma^a_{cd} g^{bd} + \Gamma^b_{cd} g^{ad} \tag{A.3}$$

Therefore

$$\begin{aligned}
\nabla_c g^{ab} &= \partial_c g^{ab} + \frac{1}{2} g^{bd} g^{ae}\Big(\partial_c g_{de} + \partial_d g_{ce} - \partial_e g_{cd}\Big) + \frac{1}{2} g^{ad} g^{be}\Big(\partial_c g_{de} + \partial_d g_{ce} - \partial_e g_{cd}\Big) \\
&= \partial_c g^{ab} + \frac{1}{2} g^{ae} g^{bd} \partial_c g_{de} + \frac{1}{2} g^{bd} g^{ae} \partial_d g_{ce} - \frac{1}{2} g^{ae} g^{bd} \partial_e g_{cd} \\
&\quad + \frac{1}{2} g^{be} g^{ad} \partial_c g_{de} + \frac{1}{2} g^{ad} g^{be} \partial_d g_{ce} - \frac{1}{2} g^{be} g^{ad} \partial_e g_{cd} \\
&= \partial_c g^{ab} + \frac{1}{2} g^{ae}\Big[\partial_c\Big(g_{de} g^{bd}\Big) - g_{de}\partial_c g^{bd}\Big] + \frac{1}{2} g^{bd}\Big[\partial_d\Big(g_{ce} g^{ae}\Big) - g_{ce}\partial_d g^{ae}\Big] \\
&\quad - \frac{1}{2} g^{ae}\Big[\partial_e\Big(g_{cd} g^{bd}\Big) - g_{cd}\partial_e g^{bd}\Big] + \frac{1}{2} g^{be}\Big[\partial_c\Big(g_{de} g^{ad}\Big) - g_{de}\partial_c g^{ad}\Big] \\
&\quad + \frac{1}{2} g^{ad}\Big[\partial_d\Big(g_{ce} g^{be}\Big) - g_{ce}\partial_d g^{be}\Big] - \frac{1}{2} g^{be}\Big[\partial_e\Big(g_{cd} g^{ad}\Big) - g_{cd}\partial_e g^{ad}\Big] \\
&= \partial_c g^{ab} + \frac{1}{2} g^{ae}\Big(\partial_c \delta^b_e - g_{de}\partial_c g^{bd}\Big) + \frac{1}{2} g^{bd}\Big(\partial_d \delta^a_c - g_{ce}\partial_d g^{ae}\Big) \\
&\quad - \frac{1}{2} g^{ae}\Big(\partial_e \delta^b_c - g_{cd}\partial_e g^{bd}\Big) + \frac{1}{2} g^{be}\Big(\partial_c \delta^a_e - g_{de}\partial_c g^{ad}\Big) \\
&\quad + \frac{1}{2} g^{ad}\Big(\partial_d \delta^b_c - g_{ce}\partial_d g^{be}\Big) - \frac{1}{2} g^{be}\Big(\partial_e \delta^a_c - g_{cd}\partial_e g^{ad}\Big)
\end{aligned}$$

δ^a_c are constant then their partial derivatives vanish. Therefore we get

$$\begin{aligned}
\nabla_c g^{ab} &= \partial_c g^{ab} - \frac{1}{2} g^{ae} g_{de}\partial_c g^{bd} - \frac{1}{2} g^{bd} g_{ce}\partial_d g^{ae} + \frac{1}{2} g^{ae} g_{cd}\partial_e g^{bd} - \frac{1}{2} g^{be} g_{de}\partial_c g^{ad} \\
&\quad - \frac{1}{2} g^{ad} g_{ce}\partial_d g^{be} + \frac{1}{2} g^{be} g_{cd}\partial_e g^{ad} \\
&= \partial_c g^{ab} - \frac{1}{2}\delta^a_d \partial_c g^{bd} - \frac{1}{2} g^{bd} g_{ce}\partial_d g^{ae} + \frac{1}{2} g^{ae} g_{cd}\partial_e g^{bd} - \frac{1}{2}\delta^b_d \partial_c g^{ad} \\
&\quad - \frac{1}{2} g^{ad} g_{ce}\partial_d g^{be} + \frac{1}{2} g^{be} g_{cd}\partial_e g^{ad} \\
&= \partial_c g^{ab} - \frac{1}{2}\partial_c g^{ba} - \frac{1}{2} g^{bd} g_{ce}\partial_d g^{ae} + \frac{1}{2} g^{ae} g_{cd}\partial_e g^{bd} - \frac{1}{2}\partial_c g^{ab} \\
&\quad - \frac{1}{2} g^{ad} g_{ce}\partial_d g^{be} + \frac{1}{2} g^{be} g_{cd}\partial_e g^{ad}.
\end{aligned}$$

Interchanging dummy indices d and e, we get

$$\begin{aligned}
\nabla_c g^{ab} &= -\frac{1}{2} g^{be} g_{cd}\partial_e g^{ad} + \frac{1}{2} g^{ad} g_{ce}\partial_d g^{be} - \frac{1}{2} g^{ad} g_{ce}\partial_d g^{be} + \frac{1}{2} g^{be} g_{cd}\partial_e g^{ad} \\
&= 0. \qquad \text{(A.4)}
\end{aligned}$$

A.3 Covariant derivative for scalar field, $\nabla_a \phi$

We will show that covariant derivative for scalar field is just ordinary derivative. Considering

$$\begin{aligned}
\nabla_a \phi &= \nabla_a \big(X_b X^b\big) \\
&= X_b \nabla_a X^b + X^b \nabla_a X_b \\
&= 2X_b \partial_a X^b + 2X_b \Gamma^b_{ac} X^c \\
&= 2\partial_a \big(X_b X^b\big) - 2X^b \partial_a X_b + X_b X^c g^{bd} \big(\partial_a g_{dc} + \partial_c g_{ad} - \partial_d g_{ac}\big) \\
&= 2\partial_a \phi - 2X^b \partial_a X_b + X^c X^d \partial_a g_{dc} + X^c X^d \partial_c g_{ad} - X^c X^d \partial_d g_{ac}.
\end{aligned}$$

Interchanging dummy indices c and d yields that the last and the forth term cancel out. Therefore we get

$$\begin{aligned}
\nabla_a \phi &= 2\partial_a \phi - 2X^b \partial_a X_b + X^c X^d \partial_a g_{dc} \\
&= 2\partial_a \phi - 2X^b \partial_a X_b + X^c \partial_a \big(g_{dc} X^d\big) - X^c g_{dc} \partial_a X^d \\
&= 2\partial_a \phi - 2X^b \partial_a X_b + X^c \partial_a X_c - X_d \partial_a X^d \\
&= 2\partial_a \phi - X^b \partial_a X_b - X_d \partial_a X^d \\
&= 2\partial_a \phi - \partial_a \big(X_b X^b\big) \\
&= \partial_a \phi
\end{aligned} \tag{A.5}$$

A.4 $R^a{}_{bcd} = -R^a{}_{bdc}$

Proving the identity $R^a{}_{bcd} = -R^a{}_{bdc}$ is as follows

$$\begin{aligned}
R^a{}_{bcd} &= \partial_c \Gamma^a_{bd} - \partial_d \Gamma^a_{bc} + \Gamma^e_{bd} \Gamma^a_{ec} - \Gamma^e_{bc} \Gamma^a_{ed} \\
&= -\big(- \partial_c \Gamma^a_{bd} + \partial_d \Gamma^a_{bc} - \Gamma^e_{bd} \Gamma^a_{ec} + \Gamma^e_{bc} \Gamma^a_{ed}\big) \\
&= -\big(\partial_d \Gamma^a_{bc} - \partial_c \Gamma^a_{bd} + \Gamma^e_{bc} \Gamma^a_{ed} - \Gamma^e_{bd} \Gamma^a_{ec}\big) \\
\therefore \quad R^a{}_{bcd} &= -R^a{}_{bdc}
\end{aligned} \tag{A.6}$$

A.5 $R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0$

Proving the identity $R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} = 0$ is as follows

$$
\begin{aligned}
1. \qquad R^a{}_{bcd} &= \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} \\
2. \qquad R^a{}_{dbc} &= \partial_b\Gamma^a_{dc} - \partial_c\Gamma^a_{db} + \Gamma^e_{dc}\Gamma^a_{eb} - \Gamma^e_{db}\Gamma^a_{ec} \\
3. \qquad R^a{}_{cdb} &= \partial_d\Gamma^a_{cb} - \partial_b\Gamma^a_{cd} + \Gamma^e_{cb}\Gamma^a_{ed} - \Gamma^e_{cd}\Gamma^a_{eb}
\end{aligned}
$$

$$
\begin{aligned}
\therefore \qquad R^a{}_{bcd} + R^a{}_{dbc} + R^a{}_{cdb} &= \partial_c\Gamma^a_{bd} - \partial_d\Gamma^a_{bc} + \Gamma^e_{bd}\Gamma^a_{ec} - \Gamma^e_{bc}\Gamma^a_{ed} + \partial_b\Gamma^a_{dc} \\
&\quad -\partial_c\Gamma^a_{db} + \Gamma^e_{dc}\Gamma^a_{eb} - \Gamma^e_{db}\Gamma^a_{ec} + \partial_d\Gamma^a_{cb} \\
&\quad -\partial_b\Gamma^a_{cd} + \Gamma^e_{cb}\Gamma^a_{ed} - \Gamma^e_{cd}\Gamma^a_{eb} \\
&= 0 \qquad \text{(A.7)}
\end{aligned}
$$

A.6 Bianchi identities

Considering covariant derivative of Riemann tensor evaluated in locally inertial coordinate:

$$\nabla_{\hat{a}}R_{\hat{b}\hat{c}\hat{d}\hat{e}} = \partial_{\hat{a}}R_{\hat{b}\hat{c}\hat{d}\hat{e}} \qquad \text{(A.8)}$$

Hats on their indices represent locally inertial coordinate. Notice that for locally inertial coordinate, Cristoffel symbols which contain first-order derivatives of metrics must vanish at any points. But the second derivatives of metrics in Riemann tensor do not vanish. Therefore

$$
\begin{aligned}
R_{\hat{b}\hat{c}\hat{d}\hat{e}} &= g_{\hat{a}\hat{b}}R^{\hat{a}}{}_{\hat{c}\hat{d}\hat{e}} \\
&= g_{\hat{a}\hat{b}}\left(\partial_{\hat{d}}\Gamma^{\hat{a}}_{\hat{c}\hat{e}} - \partial_{\hat{e}}\Gamma^{\hat{a}}_{\hat{d}\hat{c}}\right) \\
&= g_{\hat{a}\hat{b}}g^{\hat{a}\hat{f}}\left(\partial_{\hat{d}}\partial_{\hat{c}}g_{\hat{e}\hat{f}} + \partial_{\hat{d}}\partial_{\hat{e}}g_{\hat{c}\hat{f}} - \partial_{\hat{d}}\partial_{\hat{f}}g_{\hat{c}\hat{e}} - \partial_{\hat{e}}\partial_{\hat{d}}g_{\hat{c}\hat{f}} - \partial_{\hat{e}}\partial_{\hat{c}}g_{\hat{d}\hat{f}} + \partial_{\hat{e}}\partial_{\hat{f}}g_{\hat{d}\hat{c}}\right) \\
\therefore \qquad R_{\hat{b}\hat{c}\hat{d}\hat{e}} &= \frac{1}{2}\left(\partial_{\hat{d}}\partial_{\hat{c}}g_{\hat{e}\hat{b}} - \partial_{\hat{d}}\partial_{\hat{b}}g_{\hat{c}\hat{e}} - \partial_{\hat{e}}\partial_{\hat{c}}g_{\hat{d}\hat{b}} + \partial_{\hat{e}}\partial_{\hat{b}}g_{\hat{d}\hat{c}}\right). \qquad \text{(A.9)}
\end{aligned}
$$

We therefore obtain

$$
\begin{aligned}
\nabla_{\hat{a}}R_{\hat{b}\hat{c}\hat{d}\hat{e}} &+ \nabla_{\hat{c}}R_{\hat{a}\hat{b}\hat{d}\hat{e}} + \nabla_{\hat{b}}R_{\hat{c}\hat{a}\hat{d}\hat{e}} \\
&= \frac{1}{2}\left(\partial_{\hat{a}}\partial_{\hat{d}}\partial_{\hat{c}}g_{\hat{e}\hat{b}} - \partial_{\hat{a}}\partial_{\hat{d}}\partial_{\hat{b}}g_{\hat{c}\hat{e}} - \partial_{\hat{a}}\partial_{\hat{e}}\partial_{\hat{c}}g_{\hat{d}\hat{b}} + \partial_{\hat{a}}\partial_{\hat{e}}\partial_{\hat{b}}g_{\hat{d}\hat{c}}\right. \\
&\quad +\partial_{\hat{c}}\partial_{\hat{d}}\partial_{\hat{b}}g_{\hat{e}\hat{a}} - \partial_{\hat{c}}\partial_{\hat{d}}\partial_{\hat{a}}g_{\hat{b}\hat{e}} - \partial_{\hat{c}}\partial_{\hat{e}}\partial_{\hat{b}}g_{\hat{d}\hat{a}} + \partial_{\hat{c}}\partial_{\hat{e}}\partial_{\hat{a}}g_{\hat{d}\hat{b}} \\
&\quad \left.+\partial_{\hat{b}}\partial_{\hat{d}}\partial_{\hat{a}}g_{\hat{e}\hat{c}} - \partial_{\hat{b}}\partial_{\hat{d}}\partial_{\hat{c}}g_{\hat{a}\hat{e}} - \partial_{\hat{b}}\partial_{\hat{e}}\partial_{\hat{a}}g_{\hat{d}\hat{c}} + \partial_{\hat{b}}\partial_{\hat{e}}\partial_{\hat{c}}g_{\hat{d}\hat{a}}\right) \\
&= 0. \qquad \text{(A.10)}
\end{aligned}
$$

A.7 Conservation of Einstein tensor: $\nabla^b G_{ab} = 0$

Considering Bianchi identities,

$$\begin{aligned}
\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} &= 0 \\
g^{ae} g^{bc} \big(\nabla_e R_{abcd} + \nabla_d R_{abec} + \nabla_c R_{abde} \big) &= 0 \\
\nabla^a R_{ad} - \nabla_d R + \nabla^b R_{bd} &= 0.
\end{aligned}$$

Renaming indices b to a, we have

$$\begin{aligned}
2\nabla^a R_{ad} - \nabla_d R &= 0 \\
\nabla^a R_{ad} - \frac{1}{2} \nabla_d R &= 0 \\
\nabla^a \big(R_{ad} - \frac{1}{2} g_{ad} R \big) &= 0.
\end{aligned}$$

We then see that twice-contracted Bianchi identities is equivalent to

$$\nabla^a G_{ad} = 0. \tag{A.11}$$

Appendix B

Detail calculation

B.1 Variance of electromagnetic wave equation under Galilean transformation

Consider electromagnetic wave equation

$$\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}+\frac{\partial^2\phi}{\partial z^2}-\frac{1}{c^2}\frac{\partial^2\phi}{\partial t^2} = 0. \tag{B.1}$$

Transforming $t \rightarrow t'$,

$$\frac{\partial\phi}{\partial t}=\frac{\partial t'}{\partial t}\frac{\partial\phi}{\partial t'}+\frac{\partial x'}{\partial t}\frac{\partial\phi}{\partial x'}+\frac{\partial y'}{\partial t}\frac{\partial\phi}{\partial y'}+\frac{\partial z'}{\partial t}\frac{\partial\phi}{\partial z'}.$$

Using $x' = x - vt, y' = y, z' = z, t' = t$, we get

$$\begin{aligned}
\frac{\partial\phi}{\partial t} &= \frac{\partial\phi}{\partial t'}-v\frac{\partial\phi}{\partial x'}\\
\frac{\partial^2\phi}{\partial t^2} &= \frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial t'}\right)-v\frac{\partial}{\partial t}\left(\frac{\partial\phi}{\partial x'}\right)\\
&= \left(\frac{\partial t'}{\partial t}\frac{\partial^2\phi}{\partial t'^2}+\frac{\partial x'}{\partial t}\frac{\partial^2\phi}{\partial x'\partial t'}+\frac{\partial y'}{\partial t}\frac{\partial^2\phi}{\partial y'\partial t'}+\frac{\partial z'}{\partial t}\frac{\partial^2\phi}{\partial z'\partial t'}\right)\\
&\quad -v\left(\frac{\partial t'}{\partial t}\frac{\partial^2\phi}{\partial t'\partial x'}+\frac{\partial x'}{\partial t}\frac{\partial^2\phi}{\partial x'^2}+\frac{\partial y'}{\partial t}\frac{\partial^2\phi}{\partial y'\partial t'}+\frac{\partial z'}{\partial t}\frac{\partial^2\phi}{\partial z'\partial t'}\right)\\
&= \frac{\partial^2\phi}{\partial t'^2}-2v\frac{\partial^2\phi}{\partial x'\partial t'}+v^2\frac{\partial^2\phi}{\partial x'^2}.
\end{aligned} \tag{B.2}$$

Transforming $x \to x'$

$$\begin{aligned}\frac{\partial \phi}{\partial x} &= \frac{\partial t'}{\partial x}\frac{\partial \phi}{\partial t'} + \frac{\partial x'}{\partial x}\frac{\partial \phi}{\partial x'} + \frac{\partial y'}{\partial x}\frac{\partial \phi}{\partial y'} + \frac{\partial z'}{\partial x}\frac{\partial \phi}{\partial z'} \\ &= \frac{\partial \phi}{\partial x'} \\ \frac{\partial^2 \phi}{\partial x^2} &= \frac{\partial^2 \phi}{\partial x'^2} \end{aligned} \qquad \text{(B.3)}$$

The y-axis and z-axis do not change, we have

$$\frac{\partial^2 \phi}{\partial y^2} = \frac{\partial^2 \phi}{\partial y'^2} \qquad \text{(B.4)}$$

and

$$\frac{\partial^2 \phi}{\partial z^2} = \frac{\partial^2 \phi}{\partial z'^2}. \qquad \text{(B.5)}$$

Substituting all equations of transformation into equations (B.1),

$$\frac{\partial^2 \phi}{\partial x'^2} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2} + \frac{2v}{c^2}\frac{\partial^2 \phi}{\partial x' \partial t'} - \frac{v^2}{c^2}\frac{\partial^2 \phi}{\partial x'^2},$$

we finally have equation (2.5):

$$\frac{c^2 - v^2}{c^2}\frac{\partial^2 \phi}{\partial x'^2} + \frac{2v}{c^2}\frac{\partial^2 \phi}{\partial t' \partial x'} + \frac{\partial^2 \phi}{\partial y'^2} + \frac{\partial^2 \phi}{\partial z'^2} - \frac{1}{c^2}\frac{\partial^2 \phi}{\partial t'^2} = 0 \qquad \text{(B.6)}$$

B.2 Poisson's equation for Newtonian gravitational field

Considering a close surface S enclosing a mass M. The quantity $\mathbf{g} \cdot \mathbf{n} \mathrm{d}S$ is defined as the gravitational flux passing through the surface element $\mathrm{d}S$ where $\mathbf{n}$ is the outward unit vector normal to S. The total gravitational flux through S is then given by

$$\int_S \mathbf{g} \cdot \mathbf{n} \mathrm{d}S = -GM \int_S \frac{\mathbf{e}_r \cdot \mathbf{n}}{r^2} \mathrm{d}S. \qquad \text{(B.7)}$$

By definition $(\mathbf{e}_r \cdot \mathbf{n}/r^2 \mathrm{d}S) = \cos\theta \; \mathrm{d}S/r^2$ is the element of solid angle $\mathrm{d}\Omega$ subtended at M by the element surface $\mathrm{d}S$. Therefore equation (B.7) becomes

$$\begin{aligned}\int_S \mathbf{g} \cdot \mathbf{n} \mathrm{d}S &= -GM \int \mathrm{d}\Omega = -4\pi GM. \\ &= -4\pi G \int_{\mathrm{v}} \rho(\mathbf{r}) \mathrm{d}V \end{aligned}$$

From application of divergence theorem, the left side of this equation can be rewritten

$$\int_S \mathbf{g} \cdot \mathbf{n} \mathrm{d}S = \int_{\mathrm{v}} \nabla \cdot \mathbf{g} \mathrm{d}V. \tag{B.8}$$

Therefore we obtain

$$\nabla \cdot \mathbf{g} = -4\pi G\rho.$$

Recall that

$$\mathbf{g} = -\nabla\Phi. \tag{B.9}$$

We therefore have result as in equation (3.57),

$$\nabla^2\Phi = 4\pi G\rho.$$

B.3 Variation of Cristoffel symbols : $\delta\Gamma^a_{bc}$

From equation (3.14) we have

$$\begin{aligned}
\Gamma^a_{bc} &= \frac{1}{2}g^{ad}\big(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\big) \\
\delta\Gamma^a_{bc} &= \frac{1}{2}\delta g^{ad}\big(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\big) + \frac{1}{2}g^{ad}\big(\partial_b \delta g_{dc} + \partial_c \delta g_{bd} - \partial_d \delta g_{bc}\big).
\end{aligned}$$

Considering variation of g^{ad},

$$\begin{aligned}
\delta g^{ad} &= \delta_f{}^d \delta_d{}^f \delta g^{ad} \\
&= \delta_f{}^d g_{de} g^{ef} \delta g^{ad} \\
&= \delta_f{}^d g^{ef}\Big[\delta\big(g_{de}g^{ad}\big) - g^{ad}\delta g_{de}\Big] \\
&= \delta_f{}^d g^{ef}\Big[\delta(\delta_e{}^a) - g^{ad}\delta g_{de}\Big] \\
&= -g^{ed}g^{ad}\delta g_{de},
\end{aligned}$$

therefore

$$\begin{aligned}
\delta\Gamma^a_{bc} &= -\frac{1}{2}g^{ed}g^{ad}\delta g_{de}\Big(\partial_b g_{dc} + \partial_c g_{bd} - \partial_d g_{bc}\Big) + \frac{1}{2}g^{ad}\Big(\partial_b \delta g_{dc} + \partial_c \delta g_{bd} - \partial_d \delta g_{bc}\Big). \\
&= -g^{ad}\Gamma^e_{bc}\delta g_{ed} + \frac{1}{2}g^{ad}\Big(\partial_b \delta g_{dc} + \partial_c \delta g_{bd} - \partial_d \delta g_{bc}\Big) \\
&= \frac{1}{2}g^{ad}\Big(\partial_b \delta g_{dc} + \partial_c \delta g_{bd} - \partial_d \delta g_{bc} - 2\Gamma^e_{bc}\delta g_{ed}\Big) \\
&= \frac{1}{2}g^{ad}\Big(\partial_b \delta g_{dc} - \Gamma^e_{bc}\delta g_{ed} - \Gamma^e_{bd}\delta g_{ec} + \partial_c \delta g_{bd} - \Gamma^e_{bc}\delta g_{ed} - \Gamma^e_{dc}\delta g_{eb} \\
&\quad -\partial_d \delta g_{bc} + \Gamma^e_{bd}\delta g_{ec} + \Gamma^e_{dc}\delta g_{eb}\Big)
\end{aligned}$$

$$\therefore \quad \delta\Gamma^a_{bc} = \frac{1}{2}g^{ad}\big(\nabla_b \delta g_{dc} + \nabla_c \delta g_{bd} - \nabla_d \delta g_{bc}\big). \tag{B.10}$$

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